8.2: The Inverse Laplace Transform

In Section 8.1 we defined the Laplace transform of \( f(t) \) by

\[
\mathcal{L}(f) = \int_0^\infty e^{-st} f(t) \, dt.
\]
We’ll also say that \( f \) is an inverse Laplace Transform of \( F \), and write

\[
\{ f = \mathcal{L}^{-1}(F). \}
\]

To solve differential equations with the Laplace transform, we must be able to obtain \( f \) from its transform \( F \). There’s a formula for doing this, but we can’t use it because it requires the theory of functions of a complex variable. Fortunately, we can use the table of Laplace transforms to find inverse transforms that we’ll need.

Example \( \PageIndex{1} \):

Use the table of Laplace transforms to find

\[
\text{a. } \mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right)
\]

\[
\text{b. } \mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right).
\]

Solution a

Setting \( b=1 \) in the transform pair

\[
\sinh bt \leftrightarrow \frac{b}{s^2-b^2}
\]

shows that

\[
\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right)=\sinh t.
\]

Solution b

Setting \( \omega=3 \) in the transform pair

\[
\cos\omega t \leftrightarrow \frac{s}{s^2+\omega^2}
\]

shows that

\[
\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right)=\cos3t.
\]

The next theorem enables us to find inverse transforms of linear combinations of transforms in the table. We omit the proof.

Theorem \( \PageIndex{1} \): Linearity Property

If \( \{ F_1, F_2, \ldots, F_n \} \) \( \text{(F_n)} \) are Laplace transforms and \( \{ c_1, c_2, \ldots, c_n \} \) are constants, then

\[
\{ \mathcal{L}^{-1}(c_1 F_1 + c_2 F_2 + \cdots + c_n F_n) = c_1 \mathcal{L}^{-1}(F_1) + c_2 \mathcal{L}^{-1}(F_2) + \cdots + c_n \mathcal{L}^{-1}(F_n) \}
\]

Example \( \PageIndex{2} \)
Find
\[\mathcal{L}^{-1}\left(\frac{8}{s+5}+\frac{7}{s^2+3}\right).\]

**Solution**

From the table of Laplace transforms in Section 8.8,
\[e^{at}\leftrightarrow \frac{1}{s-a} \quad \text{and} \quad \sin\omega t\leftrightarrow \frac{\omega}{s^2+\omega^2}.\]

Theorem \(\PageIndex{1}\) with \(a=-5\) and \(\omega=\sqrt{3}\) yields
\[
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{8}{s+5}+\frac{7}{s^2+3}\right) &= 8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right)+7\mathcal{L}^{-1}\left(\frac{1}{s^2+3}\right)
&= 8e^{-5t}+\frac{7}{\sqrt{3}}\sin\sqrt{3}t.
\end{aligned}
\]

**Example \(\PageIndex{3}\)**

Find
\[\mathcal{L}^{-1}\left(\frac{3s+8}{s^2+2s+5}\right).\]

**Solution**

Completing the square in the denominator yields
\[\frac{3s+8}{s^2+2s+5} = \frac{3s+8}{(s+1)^2+4}.\]

Because of the form of the denominator, we consider the transform pairs
\[e^{-t}\cos 2t\leftrightarrow \frac{s+1}{(s+1)^2+4} \quad \text{and} \quad e^{-t}\sin 2t\leftrightarrow \frac{2}{(s+1)^2+4},\]

and write
\[
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{3s+8}{(s+1)^2+4}\right) &= \mathcal{L}^{-1}\left(\frac{3s+3}{(s+1)^2+4}\right)+\frac{5}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+4}\right)
&= e^{-t}(3\cos 2t+\frac{5}{2}\sin 2t).
\end{aligned}
\]

**Note**

We’ll often write inverse Laplace transforms of specific functions without explicitly stating how they are obtained. In such cases you should refer to the table of Laplace transforms in Section 8.8.
Inverse Laplace Transforms of Rational Functions

Using the Laplace transform to solve differential equations often requires finding the inverse transform of a rational function

\[ F(s) = \frac{P(s)}{Q(s)}, \text{'nonumber'} \]

where \(P\) and \(Q\) are polynomials in \(s\) with no common factors. Since it can be shown that \(\lim_{s \to \infty} F(s) = 0\) if \(F\) is a Laplace transform, we need only consider the case where \(\text{degree}(P) < \text{degree}(Q)\). To obtain \(\mathcal{L}^{-1}(F)\), we find the partial fraction expansion of \(F\), obtain inverse transforms of the individual terms in the expansion from the table of Laplace transforms, and use the linearity property of the inverse transform. The next two examples illustrate this.

Example \(\PageIndex{4}\)

Find the inverse Laplace transform of

\[ F(s) = \frac{3s+2}{s^2-3s+2}. \]

**Solution**

(Method 1)

Factoring the denominator in Equation \(\ref{eq:8.2.1}\) yields

\[ F(s) = \frac{3s+2}{(s-1)(s-2)}. \]

The form for the partial fraction expansion is

\[ \frac{3s+2}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}. \]

Multiplying this by \((s-1)(s-2)\) yields

\[ 3s+2 = (s-2)A + (s-1)B. \]

Setting \((s=2)\) yields \(B=8\) and setting \((s=1)\) yields \(A=-5\). Therefore

\[ F(s) = \frac{-5}{s-1} + \frac{8}{s-2}. \]

and

\[ \mathcal{L}^{-1}(F) = -5e^t + 8e^{2t}. \]

(Method 2) We don’t really have to multiply Equation \(\ref{eq:8.2.3}\) by \((s-1)(s-2)\) to compute \(A\) and \(B\). We can obtain \(A\) by simply ignoring the factor \((s-1)\) in the denominator of Equation \(\ref{eq:8.2.2}\) and setting \((s=1)\) elsewhere;
thus,

\[
A = \left. \frac{3s+2}{s-2} \right|_{s=1} = \frac{3 \cdot 1 + 2}{1 - 2} = -5.
\]

Similarly, we can obtain \( B \) by ignoring the factor \((s-2)\) in the denominator of Equation \ref{eq:8.2.2} and setting \( s = 2 \) elsewhere; thus,

\[
B = \left. \frac{3s+2}{s-1} \right|_{s=2} = \frac{3 \cdot 2 + 2}{2 - 1} = 8.
\]

To justify this, we observe that multiplying Equation \ref{eq:8.2.3} by \( (s-1) \) yields

\[
\frac{3s+2}{s-1} - A(s-1)B = \nonumber
\]

and setting \( s = 1 \) leads to Equation \ref{eq:8.2.4}. Similarly, multiplying Equation \ref{eq:8.2.3} by \( (s-2) \) yields

\[
\frac{3s+2}{s-2} - (s-2)A + B = \nonumber
\]

and setting \( s = 2 \) leads to Equation \ref{eq:8.2.5}. (It isn’t necessary to write the last two equations. We wrote them only to justify the shortcut procedure indicated in Equation \ref{eq:8.2.4} and Equation \ref{eq:8.2.5}.)

The shortcut employed in the second solution of Example \( \PageIndex{4} \) is Heaviside’s method. The next theorem states this method formally. For a proof and an extension of this theorem, see Exercise 8.2.10.

Theorem \( \PageIndex{2} \)

Suppose

\[
F(s) = \frac{P(s)}{(s-s_1)(s-s_2)\ldots(s-s_n)},
\]

where \( s_1, s_2, \ldots, s_n \) are distinct and \( P(s) \) is a polynomial of degree less than \( n \). Then

\[
F(s) = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_n}{s-s_n}, \nonumber
\]

where \( A_i \) can be computed from Equation \ref{eq:8.2.6} by ignoring the factor \( (s-s_i) \) and setting \( s=s_i \) elsewhere.

Example \( \PageIndex{5} \)

Find the inverse Laplace transform of

\[
F(s) = \frac{6 + (s+1)(s^2-5s+11)}{s(s-1)(s-2)(s+1)}.
\]

Solution

The partial fraction expansion of Equation \ref{eq:8.2.7} is of the form

\[
F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} + \frac{D}{s+1}.
\]
To find \( A \), we ignore the factor \( s \) in the denominator of Equation \ref{eq:8.2.7} and set \( s=0 \) elsewhere. This yields

\[
A = \frac{6+(1)(11)}{(-1)(-2)(1)} = \frac{17}{2}.
\]

Similarly, the other coefficients are given by

\[
B = \frac{6+(2)(7)}{(1)(-1)(2)} = -10,
\]
\[
C = \frac{6+3(5)}{2(1)(3)} = \frac{7}{2},
\]
and
\[
D = \frac{6}{(-1)(-2)(-3)} = -1.
\]

Therefore

\[
F(s) = \frac{17}{2} \cdot \frac{1}{s} - 10 \cdot \frac{1}{s-1} + \frac{7}{2} \cdot \frac{1}{s-2} - \frac{1}{s+1}.
\]

We didn’t “multiply out” the numerator in Equation \ref{eq:8.2.7} before computing the coefficients in Equation \ref{eq:8.2.8}, since it wouldn’t simplify the computations.

Example \ref{ex:8.2.6}

Find the inverse Laplace transform of

\[
F(s) = \frac{8-(s+2)(4s+10)}{(s+1)(s+2)^2}.
\]

Solution

The form for the partial fraction expansion is

\[
F(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}.
\]

Because of the repeated factor \( (s+2)^2 \) in Equation \ref{eq:8.2.9}, Heaviside’s method doesn’t work. Instead, we find a common denominator in Equation \ref{eq:8.2.10}. This yields

\[
F(s) = \frac{A(s+2)^2 + B(s+1)(s+2) + C(s+1)}{(s+1)(s+2)^2}.
\]
If Equation \ref{eq:8.2.9} and Equation \ref{eq:8.2.11} are to be equivalent, then

\[
A(s+2)^2+B(s+1)(s+2)+C(s+1)=8-(s+2)(4s+10).
\]

The two sides of this equation are polynomials of degree two. From a theorem of algebra, they will be equal for all \(s\) if they are equal for any three distinct values of \(s\). We may determine \(A\), \(B\) and \(C\) by choosing convenient values of \(s\).

The left side of Equation \ref{eq:8.2.12} suggests that we take \(s=-2\) to obtain \(C=-8\), and \(s=-1\) to obtain \(A=2\). We can now choose any third value of \(s\) to determine \(B\). Taking \(s=0\) yields \(4A+2B+C=-12\). Since \(A=2\) and \(C=-8\) this implies that \(B=-6\). Therefore

\[
F(s)=\frac{2}{s+1}-\frac{6}{s+2}-\frac{8}{(s+2)^2}
\]

and

\[
\begin{aligned}
\mathcal{L}^{-1}(F)&= 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)-6\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)-8\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) \\
&=2e^{-t}-6e^{-2t}-8te^{-2t}.
\end{aligned}
\]

Example \(\PageIndex{7}\)

Find the inverse Laplace transform of

\[
F(s)=\frac{s^2-5s+7}{(s+2)^3}.
\]

**Solution**

The form for the partial fraction expansion is

\[
F(s)=\frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{C}{(s+2)^3}.
\]

The easiest way to obtain \(A\), \(B\), and \(C\) is to expand the numerator in powers of \((s+2)\). This yields

\[
(s^2-5s+7)=[(s+2)-2]^2-5[(s+2)-2]+7=(s+2)^2-9(s+2)+21.
\]

Therefore

\[
\begin{aligned}
F(s)&=\frac{(s+2)^2-9(s+2)+21}{(s+2)^3}\\
&=\frac{1}{s+2}-\frac{9}{(s+2)^2}+\frac{21}{(s+2)^3}
\end{aligned}
\]

and

\[
\begin{aligned}
\mathcal{L}^{-1}(F)&= \mathcal{L}^{-1}\left(\frac{1}{s+2}\right)-9\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right)+\frac{21}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+2)^3}\right) \\
&=e^{-2t}\mathcal{L}^{-1}(1-9t+\frac{21}{2}t^2)+e^{-2t}(1-9t+\frac{21}{2}t^2).\mathcal{L}^{-1}(t^2).
\end{aligned}
\]

Example \(\PageIndex{8}\)
Find the inverse Laplace transform of
\[
F(s) = \frac{1-s(5+3s)}{s[(s+1)^2+1]}.
\]

**Solution**

One form for the partial fraction expansion of \((F)\) is
\[
F(s) = \frac{A}{s} + \frac{Bs+C}{(s+1)^2+1}.
\]

However, we see from the table of Laplace transforms that the inverse transform of the second fraction on the right of Equation \ref{eq:8.2.14} will be a linear combination of the inverse transforms
\[
e^{-t}\cos t \quad \text{and} \quad e^{-t}\sin t
\]
of
\[
\frac{s+1}{(s+1)^2+1} \quad \text{and} \quad \frac{1}{(s+1)^2+1}
\]
respectively. Therefore, instead of Equation \ref{eq:8.2.14} we write
\[
F(s) = \frac{A}{s} + \frac{B(s+1)+C}{(s+1)^2+1}.
\]

Finding a common denominator yields
\[
F(s) = \frac{A[(s+1)^2+1]+B(s+1)s+Cs}{s[(s+1)^2+1]}.
\]

If Equation \ref{eq:8.2.13} and Equation \ref{eq:8.2.16} are to be equivalent, then
\[
A[(s+1)^2+1]+B(s+1)s+Cs=1-s(5+3s).
\]

This is true for all \((s)\) if it is true for three distinct values of \((s)\). Choosing \((s=0), (-1), \text{and} (1)\) yields the system
\[
\begin{aligned}
A-C &= 3 \\
5A+2B+C &= -7
\end{aligned}
\]

Solving this system yields
\[
A = \frac{1}{2}, \quad B = -\frac{7}{2}, \quad C = -\frac{5}{2}.
\]

Hence, from Equation \ref{eq:8.2.15},
\[
F(s) = \frac{1}{2s} - \frac{7}{2} \frac{s+1}{(s+1)^2+1} - \frac{5}{2} \frac{1}{(s+1)^2+1}.
\]

Therefore
\[
\begin{aligned}
\mathcal{L}^{-1}(F) &= \frac{1}{2} \mathcal{L}^{-1}\left(1 \over s\right) - \frac{7}{2} \mathcal{L}^{-1}\left(s+1 \over (s+1)^2+1\right) - \frac{5}{2} \mathcal{L}^{-1}\left(1 \over (s+1)^2+1\right) \\
&= \frac{1}{2} t - \frac{7}{2} e^{-t}\cos t - \frac{5}{2} e^{-t}\sin t
\end{aligned}
\]

\[e^{-t}\cos t \quad \text{and} \quad e^{-t}\sin t \quad \text{for} \\
\{s+1\over (s+1)^2+1\} \quad \text{and} \quad \{1\over (s+1)^2+1\}.
\]
\[
\frac{\over(s+1)^2+1\right)-\{5/\over2\}}{\text{cal L}^\{-1\}\left(1/\over(s+1)^2+1\right)}\&= \{1/\over2\}e^{-t}\cos t - \{5/\over2\}e^{-t}\sin t.\end{aligned}\]

Example \(\PageIndex{9}\):

Find the inverse Laplace transform of

\[
\label{eq:8.2.17} F(s)={8+3s/\over(s^2+1)(s^2+4)}.\]

**Solution**

The form for the partial fraction expansion is

\[
[F(s)={A+B\over s^2+1}+{C+D\over s^2+4}
\]

The coefficients \((A), (B), (C)\) and \((D)\) can be obtained by finding a common denominator and equating the resulting numerator to the numerator in Equation \ref{eq:8.2.17}. However, since there’s no first power of \((s)\) in the denominator of Equation \ref{eq:8.2.17}, there’s an easier way: the expansion of

\[
[F\_1(s)=\{1/\over(s^2+1)(s^2+4)\}
\]

can be obtained quickly by using Heaviside’s method to expand

\[
[\{1/\over(x+1)(x+4)\}={1/3}\left\{1/\over x+1}-{1/\over x+4}\right\}
\]

and then setting \((x=s^2)\) to obtain

\[
[\{1/\over(s^2+1)(s^2+4)\}={1/\over3}\left\{1/\over s^2+1}-{1/\over s^2+4}\right\}
\]

Multiplying this by \((8+3s)\) yields

\[
[F(s)={8+3s/\over(s^2+1)(s^2+4)}={1/\over3}\left\{8+3s/\over s^2+1}-{8+3s/\over s^2+4}\right\}
\]

Therefore

\[
[\{\text{cal L}^\{-1\}(F)={8/\over3}\sin t+\cos t-{4/\over3}\sin 2t-\cos 2t\}
\]

**Using Technology**

Some software packages that do symbolic algebra can find partial fraction expansions very easily. We recommend that you use such a package if one is available to you, but only after you’ve done enough partial fraction expansions on your own to master the technique.