15.4: Applications of Double Integrals

Learning Objectives

- Recognize when a function of two variables is integrable over a rectangular region.
- Recognize and use some of the properties of double integrals.
- Evaluate a double integral over a rectangular region by writing it as an iterated integral.
- Use a double integral to calculate the area of a region, volume under a surface, or average value of a function over a plane region.

In this section we investigate double integrals and show how we can use them to find the volume of a solid over a rectangular region in the xy-plane. Many of the properties of double integrals are similar to those we have already discussed for single integrals.

Volumes and Double Integrals

We begin by considering the space above a rectangular region \((R)\). Consider a continuous function \((f(x,y)\geq0)\) of two variables defined on the closed rectangle \((R)\):

\[
[R=[a,b] \times [c,d]= \{(x,y) \in \mathbb{R}^2 | \ a \leq x \leq b, \ c \leq y \leq d\} \nonumber
\]

Here \(([a,b] \times [c,d])\) denotes the Cartesian product of the two closed intervals \(([a,b])\) and \(([c,d])\). It consists of rectangular pairs \(((x,y))\) such that \((a\leq x \leq b)\) and \((c\leq y \leq d)\). The graph of \((f)\) represents a surface above the \((xy)\)-plane with equation \((z = f(x,y))\) where \((z)\) is the height of the surface at the point \(((x,y))\). Let \((S)\) be the solid that lies above \((R)\) and under the graph of \((f)\) (Figure \((\PageIndex{1})\)). The base of the solid is the rectangle \((R)\) in the \((xy)\)-plane. We want
to find the volume \( V \) of the solid \( S \).

Figure \( \PageIndex{1} \): The graph of \( f(x,y) \) over the rectangle \( R \) in the \( xy \)-plane is a curved surface.

We divide the region \( R \) into small rectangles \( R_{ij} \), each with area \( \Delta A \) and with sides \( \Delta x \) and \( \Delta y \) (Figure \( \PageIndex{2} \)). We do this by dividing the interval \([a,b]\) into \( m \) subintervals and dividing the interval \([c,d]\) into \( n \) subintervals. Hence \( \Delta x = \frac{b - a}{m} \), \( \Delta y = \frac{d - c}{n} \), and \( \Delta A = \Delta x \Delta y \).

Figure \( \PageIndex{2} \): Rectangle \( R \) is divided into small rectangles \( R_{ij} \) each with area \( \Delta A \).

The volume of a thin rectangular box above \( R_{ij} \) is \( f(x_{ij}^*, y_{ij}^*) \Delta A \), where \( (x_{ij}^*, y_{ij}^*) \) is an arbitrary sample point in each \( R_{ij} \) as shown in the following figure, \( f(x_{ij}^*, y_{ij}^*) \) is the height of the corresponding thin rectangular box, and \( \Delta A \) is the area of each rectangle \( R_{ij} \).
Using the same idea for all the subrectangles, we obtain an approximate volume of the solid $S$ as

$$V \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A.$$ 

This sum is known as a **double Riemann sum** and can be used to approximate the value of the volume of the solid. Here the double sum means that for each subrectangle we evaluate the function at the chosen point, multiply by the area of each rectangle, and then add all the results.

As we have seen in the single-variable case, we obtain a better approximation to the actual volume if $m$ and $n$ become larger.

$$V = \lim_{m,n \to \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A.$$ 

or

$$V = \lim_{\Delta x, \Delta y \to 0} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A.$$ 

Note that the sum approaches a limit in either case and the limit is the volume of the solid with the base $R$. Now we are ready to define the double integral.

**Definition**

The double integral of the function $f(x, y)$ over the rectangular region $R$ in the $(xy)$-plane is defined as

$$\iint_R f(x, y) \, dA = \lim_{m,n \to \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A.$$
If \( f(x,y) \geq 0 \), then the volume \( V \) of the solid \( S \), which lies above \( R \) in the \( xy \)-plane and under the graph of \( f \), is the double integral of the function \( f(x,y) \) over the rectangle \( R \). If the function is ever negative, then the double integral can be considered a “signed” volume in a manner similar to the way we defined net signed area in The Definite Integral.

Example \( \PageIndex{1} \): Setting up a Double Integral and Approximating It by Double Sums

Consider the function \( z = f(x, y) = 3x^2 - y \) over the rectangular region \( R = [0, 2] \times [0, 2] \) (Figure \( \PageIndex{4} \)).

a. Set up a double integral for finding the value of the signed volume of the solid \( S \) that lies above \( R \) and “under” the graph of \( f \).

b. Divide \( R \) into four squares with \( m = n = 2 \), and choose the sample point as the upper right corner point of each square: \((1,1),(2,1),(1,2), \text{ and } (2,2) \) (Figure \( \PageIndex{4} \)) to approximate the signed volume of the solid \( S \) that lies above \( R \) and “under” the graph of \( f \).

c. Divide \( R \) into four squares with \( m = n = 2 \), and choose the sample point as the midpoint of each square: \((1/2, 1/2), (3/2, 1/2), (1/2,3/2), \text{ and } (3/2, 3/2) \) to approximate the signed volume.

\( z = f(x, y) = 3x^2 - y \)

\( x \quad 0.0 \quad 1.0 \quad 1.5 \quad 2.0 \)
\( y \quad 0.0 \quad 1.0 \quad 1.5 \quad 2.0 \)
\( z \quad 0.0 \quad 5.0 \quad 10.0 \quad 15.0 \)

Figure \( \PageIndex{4} \): The function \( z=f(x,y) \) graphed over the rectangular region \( R=[0,2] \times [0,2] \).

Solution

a. As we can see, the function \( z = f(x, y) = 3x^2 - y \) is above the plane. To find the signed volume of \( S \), we need to divide the region \( R \) into small rectangles \( R_{ij} \), each with area \( \Delta A \) and with sides \( \Delta x \) and \( \Delta y \), and choose \( ((x_{ij}^*, y_{ij}^*)) \) as sample points in each \( R_{ij} \). Hence, a double integral is set up as

\[
\begin{align*}
V = \int \int_R (3x^2 - y) \, dA &= \lim_{m,n \to \infty} \sum_{i=1}^m \sum_{j=1}^n \left[ 3(x_{ij}^*)^2 - y_{ij}^* \right] \Delta A.
\end{align*}
\]

b. Approximating the signed volume using a Riemann sum with \( m = n = 2 \) we have \( \Delta A = \Delta x \Delta y = 1 \times 1 = 1 \). Also, the sample points are \((1, 1), (2, 1), (1, 2), \text{ and } (2, 2) \) as shown in the following figure.
Choose upper right corners of squares

Each square has area $\Delta A = 1$

Figure (PageIndex(5)): Subrectangles for the rectangular region $R=[0,2] \times [0,2]$. 

Hence,

$$
\begin{align*}
V &\approx \sum_{i=1}^2 \sum_{j=1}^2 f(x_{ij}^*, y_{ij}^*) \Delta A \\
&= \sum_{i=1}^2 (f(x_{i1}^*, y_{i1}^*) + f(x_{i2}^*, y_{i2}^*)) \Delta A \\
&= f(x_{11}^*, y_{11}^*) \Delta A + f(x_{21}^*, y_{21}^*) \Delta A + f(x_{12}^*, y_{12}^*) \Delta A + f(x_{22}^*, y_{22}^*) \Delta A \\
&= f(1,1)(1) + f(2,1)(1) + f(1,2)(1) + f(2,2)(1) \\
&= (3 - 1)(1) + (12 - 1)(1) + (3 - 2)(1) + (12 - 2)(1) \\
&= 2 + 11 + 1 + 10 = 24.
\end{align*}
$$

c. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have $\Delta A = \Delta x \cdot \Delta y = 1$. In this case the sample points are $(1/2, 1/2), (3/2, 1/2), (1/2, 3/2), \text{ and } (3/2, 3/2)$. Hence,

$$
\begin{align*}
V &\approx \sum_{i=1}^2 \sum_{j=1}^2 f(x_{ij}^*, y_{ij}^*) \Delta A \\
&= f(x_{11}^*, y_{11}^*) \Delta A + f(x_{21}^*, y_{21}^*) \Delta A + f(x_{12}^*, y_{12}^*) \Delta A + f(x_{22}^*, y_{22}^*) \Delta A \\
&= f(1/2,1/2)(1) + f(3/2,1/2)(1) + f(1/2,3/2)(1) + f(3/2,3/2)(1) \\
&= \frac{2}{4} + \frac{25}{4} + \left(-\frac{3}{4}\right) + \frac{21}{4} = \frac{45}{4} = 11.
\end{align*}
$$

Analysis
Notice that the approximate answers differ due to the choices of the sample points. In either case, we are introducing some error because we are using only a few sample points. Thus, we need to investigate how we can achieve an accurate answer.

Exercise \(\PageIndex{1}\)

Use the same function \(z = f(x, y) = 3x^2 - y\) over the rectangular region \(R=[0,2] \times [0,2]\).

Divide \(R\) into the same four squares with \(m = n = 2\), and choose the sample points as the upper left corner point of each square \((0,1), (1,1), (0,2),\) and \((1,2)\) (Figure \(\PageIndex{5}\)) to approximate the signed volume of the solid \(S\) that lies above \(R\) and “under” the graph of \(f\).

**Hint**

Follow the steps of the previous example.

**Answer**

\[
V \approx \sum_{i=1}^2 \sum_{j=1}^2 f(x_{ij}^*, y_{ij}^*) \Delta A = 0 \nonumber
\]

Note that we developed the concept of double integral using a rectangular region \(R\). This concept can be extended to any general region. However, when a region is not rectangular, the subrectangles may not all fit perfectly into \(R\), particularly if the base area is curved. We examine this situation in more detail in the next section, where we study regions that are not always rectangular and subrectangles may not fit perfectly in the region \(R\). Also, the heights may not be exact if the surface \((z=f(x,y))\) is curved. However, the errors on the sides and the height where the pieces may not fit perfectly within the solid \(S\) approach 0 as \(m\) and \(n\) approach infinity. Also, the double integral of the function \((z=f(x,y))\) exists provided that the function \(f\) is not too discontinuous. If the function is bounded and continuous over \(R\) except on a finite number of smooth curves, then the double integral exists and we say that \(f\) is integrable over \(R\).

Since \(\Delta A = \Delta x \Delta y = \Delta y \Delta x\), we can express \((dA)\) as \((dx, dy)\) or \((dy, dx)\). This means that, when we are using rectangular coordinates, the double integral over a region \(R\) denoted by

\[
\iint_R f(x,y) \, dA \nonumber
\]

can be written as

\[
\iint_R f(x,y) \, dx \, dy \nonumber
\]
or

\[
\iint_R f(x,y) \, dy \, dx \nonumber
\]

Now let’s list some of the properties that can be helpful to compute double integrals.
Properties of Double Integrals

The properties of double integrals are very helpful when computing them or otherwise working with them. We list here six properties of double integrals. Properties 1 and 2 are referred to as the linearity of the integral, property 3 is the additivity of the integral, property 4 is the monotonicity of the integral, and property 5 is used to find the bounds of the integral. Property 6 is used if \( f(x,y) \) is a product of two functions \( g(x) \) and \( h(y) \).

Theorem: PROPERTIES OF DOUBLE INTEGRALS

Assume that the functions \( f(x,y) \) and \( g(x,y) \) are integrable over the rectangular region \( R \); \( S \) and \( T \) are subregions of \( R \); and assume that \( m \) and \( M \) are real numbers.

1. The sum \( f(x,y)+g(x,y) \) is integrable and
   \[
   \iint_R [f(x, y) + g(x, y)],dA = \iint_R f(x,y),dA + \iint_R g(x, y),dA. \nonumber
   \]
2. If \( c \) is a constant, then \( cf(x,y) \) is integrable and
   \[
   \iint_R cf(x,y),dA = c \iint_R f(x,y),dA. \nonumber
   \]
3. If \( R=S\cup T \) and \( (S\cap T=\emptyset) \) except an overlap on the boundaries, then
   \[
   \iint_R f(x,y),dA = \iint_S f(x,y),dA + \iint_T f(x,y),dA. \nonumber
   \]
4. If \( f(x,y) \geq g(x,y) \) for \((x,y)\) in \( R \), then
   \[
   \iint_R f(x,y),dA \geq \iint_R g(x,y),dA. \nonumber
   \]
5. If \( m \leq f(x,y) \leq M \) and \( A(R) = \text{the area of } R \), then
   \[
   m \cdot A(R) \leq \iint_R f(x,y),dA \leq M \cdot A(R). \nonumber
   \]
6. In the case where \( f(x,y) \) can be factored as a product of a function \( g(x) \) of \((x)\) only and a function \( h(y) \) of \((y)\) only, then over the region \( R = \{(x,y) | a \leq x \leq b, c \leq y \leq d \} \), the double integral can be written as
   \[
   \iint_R f(x,y),dA = \left(\int_a^b g(x),dx \right) \left(\int_c^d h(y),dy \right). \nonumber
   \]

These properties are used in the evaluation of double integrals, as we will see later. We will become skilled in using these properties once we become familiar with the computational tools of double integrals. So let’s get to that now.

Iterated Integrals

So far, we have seen how to set up a double integral and how to obtain an approximate value for it. We can also imagine that evaluating double integrals by using the definition can be a very lengthy process if we choose larger values for \( m \) and \( n \). Therefore, we need a practical and convenient technique for computing double integrals. In other words, we need to learn
how to compute double integrals without employing the definition that uses limits and double sums.

The basic idea is that the evaluation becomes easier if we can break a double integral into single integrals by integrating first with respect to one variable and then with respect to the other. The key tool we need is called an iterated integral.

Definitions: iterated integrals

Assume \((a), (b), (c),\) and \((d)\) are real numbers. We define an iterated integral for a function \((f(x,y))\) over the rectangular region \(R = [a,b] \times [c,d]\) as

\[
\int_a^b \int_c^d f(x,y) \, dy \, dx = \int_a^b \left[ \int_c^d f(x,y) \, dy \right] \, dx
\]

or

\[
\int_c^d \int_a^b f(x,y) \, dx \, dy = \int_c^d \left[ \int_a^b f(x,y) \, dx \right] \, dy.
\]

The notation \(\int_a^b \left[ \int_c^d f(x,y) \, dy \right] \, dx\) means that we integrate \((f(x,y))\) with respect to \((y)\) while holding \((x)\) constant. Similarly, the notation \(\int_c^d \left[ \int_a^b f(x,y) \, dx \right] \, dy\) means that we integrate \((f(x,y))\) with respect to \((x)\) while holding \((y)\) constant. The fact that double integrals can be split into iterated integrals is expressed in Fubini’s theorem. Think of this theorem as an essential tool for evaluating double integrals.

Theorem: FUBINI’S THEOREM

Suppose that \(f(x,y)\) is a function of two variables that is continuous over a rectangular region \(R = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}\). Then we see from Figure \((\PageIndex{6})\) that the double integral of \(f\) over the region equals an iterated integral,

\[
\int_{R} f(x,y) \, dA = \int_{x=a}^{b} \left[ \int_{y=c}^{d} f(x,y) \, dy \right] \, dx = \int_{y=c}^{d} \left[ \int_{x=a}^{b} f(x,y) \, dx \right] \, dy.
\]

More generally, Fubini’s theorem is true if \(f\) is bounded on \(R\) and \(f\) is discontinuous only on a finite number of continuous curves. In other words, \(f\) has to be integrable over \(R\).

Figure \((\PageIndex{6})\): (a) Integrating first with respect to \((y)\) and then with respect to \((x)\) to find the area \(A(x)\) and then the volume \(V\); (b) integrating first with respect to \((x)\) and then with respect to \((y)\) to find the area \(A(y)\) and then the volume \(V\).
Example \((\PageIndex{2})\): Using Fubini’s Theorem

Use Fubini’s theorem to compute the double integral \(\displaystyle \iint_R f(x,y) \,dA\) where \(f(x,y) = x\) and \(R = [0, 2] \times [0, 1]\).

**Solution**

Fubini’s theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region \((R)\) become the upper and lower limits of integration.

\[
\begin{align*}
\iint_R f(x,y) \,dA &= \iint_R f(x,y) \,dx \, dy \\
&= \int_{y=0}^{y=1} \left( \int_{x=0}^{x=2} x \, dx \right) \, dy \\
&= \int_{y=0}^{y=1} \left[ \frac{x^2}{2} \right]_{x=0}^{x=2} \, dy \\
&= \int_{y=0}^{y=1} 2 \, dy = 2y \bigg|_{y=0}^{y=1} = 2
\end{align*}
\]

The double integration in this example is simple enough to use Fubini’s theorem directly, allowing us to convert a double integral into an iterated integral. Consequently, we are now ready to convert all double integrals to iterated integrals and demonstrate how the properties listed earlier can help us evaluate double integrals when the function \(f(x,y)\) is more complex. Note that the order of integration can be changed (see Example 7).

Example \((\PageIndex{3})\): Illustrating Properties i and ii

Evaluate the double integral \(\iint_R (xy - 3xy^2) \,dA, \) where \(R = \{(x,y) \,| \, 0 \leq x \leq 2, \, 1 \leq y \leq 2\}\).

**Solution**

This function has two pieces: one piece is \((xy)\) and the other is \((3xy^2)\). Also, the second piece has a constant 3. Notice how we use properties i and ii to help evaluate the double integral.

\[
\begin{align*}
\iint_R (xy - 3xy^2) \,dA &= \iint_R xy \, dA + \iint_R (-3xy^2) \,dA & \text{Property i: Integral of a sum is the sum of the integrals.} \\
&= \left( \int_{x=0}^{x=2} x \, dx \right) \left( \int_{y=1}^{y=2} dy \right) - \left( \int_{x=0}^{x=2} 3x \, dx \right) \left( \int_{y=1}^{y=2} y^2 \, dy \right) & \text{Convert double integrals to iterated integrals.} \\
&= \int_{x=0}^{x=2} x \, dx \left( \int_{y=1}^{y=2} dy \right) - 3\int_{x=0}^{x=2} x \, dx \left( \int_{y=1}^{y=2} y^2 \, dy \right) & \text{Integrate with respect to x, holding y constant.} \\
&= \left( \frac{x^2}{2} \right)_{x=0}^{x=2} \left( y^2 \right)_{y=1}^{y=2} - 3\left( \frac{x^2}{2} \right)_{x=0}^{x=2} \left( \frac{y^3}{3} \right)_{y=1}^{y=2} & \text{Integrate with respect to y.} \\
&= 2 - 2(8) - 6 \cdot 2(7) & \text{Property ii: Placing the constant before the integral.} \\
&= 3 - 28 - 6 - 14 = 3 - 44 = -41.
\end{align*}
\]
Example \(\PageIndex{4}\): Illustrating Property v.

Over the region \(R = \{(x,y) \mid 1 \leq x \leq 3, \ 1 \leq y \leq 2\}\), we have \(2 \leq x^2 + y^2 \leq 13\). Find a lower and an upper bound for the integral \(\iint_R (x^2 + y^2) \, dA\).

**Solution**

For a lower bound, integrate the constant function 2 over the region \(R\). For an upper bound, integrate the constant function 13 over the region \(R\).

\[
\begin{align*}
\int_1^2 \int_1^3 2 \, dx \, dy &= \int_1^2 [2x]_{x=1}^{x=3} \, dy = \int_1^2 2(2) \, dy = 4y \bigg|_1^2 = 4(2 - 1) = 4 \\
\int_1^2 \int_1^3 13 \, dx \, dy &= \int_1^2 [13x]_{x=1}^{x=3} \, dy = \int_1^2 13(2) \, dy = 26y \bigg|_1^2 = 26(2 - 1) = 26.
\end{align*}
\]

Hence, we obtain \(4 \leq \iint_R (x^2 + y^2) \, dA \leq 26\).

Example \(\PageIndex{5}\): Illustrating Property vi

Evaluate the integral \(\iint_R e^y \cos x \, dA\) over the region \(R = \{(x,y) \mid 0 \leq x \leq \frac{\pi}{2}, \ 0 \leq y \leq 1\}\).

**Solution**

This is a great example for property vi because the function \(f(x,y)\) is clearly the product of two single-variable functions \(e^y\) and \(\cos x\). Thus we can split the integral into two parts and then integrate each one as a single-variable integration problem.

\[
\begin{align*}
\iint_R e^y \cos x \, dA &= \int_0^1 \int_0^{\pi/2} e^y \cos x \, dx \, dy \\
&= \left(\int_0^1 e^y \, dy\right) \left(\int_0^{\pi/2} \cos x \, dx\right) \\
&= (e^y \bigg|_0^1) (\sin x \bigg|_0^{\pi/2}) \\
&= e - 1.
\end{align*}
\]

Exercise \(\PageIndex{2}\)

a. Use the properties of the double integral and Fubini’s theorem to evaluate the integral

\[
\int_{-1}^3 \int_0^1 (3 - x + 4y) \, dy \, dx.
\]

b. Show that \(0 \leq \iint_R \sin pi x \, \cos pi y \, dA \leq \frac{1}{32}\) where \(R = \{(x,y) \mid \text{left}(0, \frac{\pi}{4}\text{right})\} \setminus \{(x,y) \mid \text{left}(\text{frac{1}{2}}\text{right})\} \text{ where } (R = \text{left}(0, \frac{\pi}{4}\text{right}) \setminus \{(x,y) \mid \text{left}(\text{frac{1}{2}}\text{right})\}.

**Hint**

Use properties i. and ii. and evaluate the iterated integral, and then use property v.
As we mentioned before, when we are using rectangular coordinates, the double integral over a region \(R\) denoted by \(\iint_R f(x,y) \, dA\) can be written as \(\iint_R f(x,y) \, dx \, dy\) or \(\iint_R f(x,y) \, dy \, dx\). The next example shows that the results are the same regardless of which order of integration we choose.

Example \(\PageIndex{6}\): Evaluating an Iterated Integral in Two Ways

Let’s return to the function \(f(x,y) = 3x^2 - y\) from Example 1, this time over the rectangular region \(R = [0,2] \times [0,3]\). Use Fubini’s theorem to evaluate \(\iint_R f(x,y) \, dA\) in two different ways:

a. First integrate with respect to \(y\) and then with respect to \(x\);

b. First integrate with respect to \(x\) and then with respect to \(y\).

Solution

Figure \(\PageIndex{6}\) shows how the calculation works in two different ways.

a. First integrate with respect to \(y\) and then integrate with respect to \(x\):

\[
\begin{align*}
\iint_R f(x,y) \, dA &= \int_{x=0}^{x=2} \int_{y=0}^{y=3} (3x^2 - y) \, dy \, dx \\
&= \int_{x=0}^{x=2} \left[ 3x^2y - \frac{y^2}{2} \right]_{y=0}^{y=3} \, dx \\
&= \int_{x=0}^{x=2} \left( 9x^2 - \frac{9}{2} \right) \, dx \\
&= 3x^3 - \frac{9}{2}x \bigg|_{x=0}^{x=2} = 15.
\end{align*}
\]

b. First integrate with respect to \(x\) and then integrate with respect to \(y\):

\[
\begin{align*}
\iint_R f(x,y) \, dA &= \int_{y=0}^{y=3} \int_{x=0}^{x=2} (3x^2 - y) \, dx \, dy \\
&= \int_{y=0}^{y=3} \left[ x^3 - xy \right]_{x=0}^{x=2} \, dy \\
&= \int_{y=0}^{y=3} (8 - 2y) \, dy \\
&= 8y - y^2 \bigg|_{y=0}^{y=3} = 15.
\end{align*}
\]

Analysis

With either order of integration, the double integral gives us an answer of \(15\). We might wish to interpret this answer as a volume in cubic units of the solid \(S\) below the function \(f(x,y) = 3x^2 - y\) over the region \(R = [0,2] \times [0,3]\). However, remember that the interpretation of a double integral as a (non-signed) volume works only when the integrand \(f\) is a nonnegative function over the base region \(R\).

Exercise \(\PageIndex{3}\)

Evaluate
\[ \int_{y=-3}^{y=2} \int_{x=3}^{x=5} (2 - 3x^2 + y^2) \,dx \, dy. \nonumber\]

**Hint**

Use Fubini’s theorem.

**Answer**

\[ -\frac{1340}{3} \]

In the next example we see that it can actually be beneficial to switch the order of integration to make the computation easier. We will come back to this idea several times in this chapter.

**Example \(\PageIndex{7}\): Switching the Order of Integration**

Consider the double integral \(\iint_R x \, \sin (xy) \, dA\) over the region \(R = \{(x,y) \mid 0 \leq x \leq \pi, 1 \leq y \leq 2\}\) (Figure \(\PageIndex{7}\)).

a. Express the double integral in two different ways.

b. Analyze whether evaluating the double integral in one way is easier than the other and why.

c. Evaluate the integral.

\(\text{Figure } \PageIndex{7}: \text{ The function } z = f(x,y) = x \sin(xy) \text{ over the rectangular region } R = [0,\pi] \times [1,2].\)
\[
\iint_R x \sin(xy) \, dA = \int_{y=1}^{y=2} \left[ \int_{x=0}^{x=\pi} x \sin(xy) \, dx \right] \, dy.
\]

However, integrating with respect to \(x\) first and then integrating with respect to \(y\) requires integration by parts for the inner integral, with \(u = x\) and \(dv = \sin(xy) \, dx\). Then \(du = dx\) and \(v = -\frac{\cos(xy)}{y}\), so

\[
\iint_R x \sin(xy) \, dA = \int_{y=1}^{y=2} \left[ -\frac{x \cos(xy)}{y} \bigg|_{x=0}^{x=\pi} + \frac{1}{y} \int_{x=0}^{x=\pi} \cos(xy) \, dx \right] \, dy.
\]

Since the evaluation is getting complicated, we will only do the computation that is easier to do, which is clearly the first method.

c. Evaluate the double integral using the easier way.

\[
\begin{align*}
\iint_R x \, \sin(xy) \, dA &= \int_{x=0}^{x=\pi} \left[ \int_{u=x}^{u=2x} \sin(u) \, du \right] \, dx \\
&= \int_{x=0}^{x=\pi} (-\cos 2x + \cos x) \, dx \\
&= \left(- \frac{1}{2} \sin 2x + \sin x\right) \bigg|_{x=0}^{x=\pi} = 0.
\end{align*}
\]

Exercise \((PageIndex\{4\})\)

Evaluate the integral \(\iint_R xe^{xy} \, dA\) where \(R = [0,1] \times [0, \ln 5]\).

\textbf{Hint}

Integrate with respect to \(y\) first.

\textbf{Answer}

\[
\frac{4 - \ln 5}{\ln 5}
\]

\textbf{Applications of Double Integrals}

Double integrals are very useful for finding the area of a region bounded by curves of functions. We describe this situation in more detail in the next section. However, if the region is a rectangular shape, we can find its area by integrating the constant function \(f(x,y) = 1\) over the region \(R\).

\textbf{Definition: area of the region}

The area of the region \(R\) is given by \(A(R) = \iint_R 1 \, dA\).
This definition makes sense because using \( f(x,y) = 1 \) and evaluating the integral make it a product of length and width.

Let’s check this formula with an example and see how this works.

**Example \( \PageIndex{8} \): Finding Area Using a Double Integral**

Find the area of the region \( \{(x,y) : 0 \leq x \leq 3, \ 0 \leq y \leq 2 \} \) by using a double integral, that is, by integrating \( 1 \) over the region \( \{(x,y)\} \).

**Solution**

The region is rectangular with length \( 3 \) and width \( 2 \), so we know that the area is \( 6 \). We get the same answer when we use a double integral:

\[
[A(R) = \int_0^2 \int_0^3 1 \, dx \, dy = \left[ x \right]_0^3 \int_0^2 dy = 3 \int_0^2 dy = 3y \bigg|_0^2 = 3(2) = 6 \, \text{units}^2.]
\]

We have already seen how double integrals can be used to find the volume of a solid bounded above by a function \( f(x,y) \geq 0 \) over a region \( \{(x,y)\} \) provided \( f(x,y) \geq 0 \) for all \( (x,y) \) in \( \{(x,y)\} \). Here is another example to illustrate this concept.

**Example \( \PageIndex{9} \): Volume of an Elliptic Paraboloid**

Find the volume \( \{(V)\} \) of the solid \( \{(S)\} \) that is bounded by the elliptic paraboloid \( (2x^2 + y^2 + z = 27) \), the planes \( \{(x = 3)\} \) and \( \{(y = 3)\} \), and the three coordinate planes.

**Solution**

First notice the graph of the surface \( (z = 27 - 2x^2 - y^2) \) in Figure \( \PageIndex{8}(a) \) and above the square region \( \{(R) = [\{-3,3]\} \times [\{-3,3]\}] \). However, we need the volume of the solid bounded by the elliptic paraboloid \( (2x^2 + y^2 + z = 27) \), the planes \( \{(x = 3)\} \) and \( \{(y = 3)\} \), and the three coordinate planes.

Now let’s look at the graph of the surface in Figure \( \PageIndex{8}(b) \). We determine the volume \( \{(V)\} \) by evaluating the double integral over \( \{(R) \} \):

\[
\begin{align*}
\int \int_R z \, dA &= \int \int_R (27 - 2x^2 - y^2) \, dA
\end{align*}
\]
\[ \int_{y=0}^{y=3} \int_{x=0}^{x=3} (27 - 2x^2 - y^2) \, dx \, dy \]

\[ = \int_{y=0}^{y=3} \left[ 27x - \frac{2}{3} x^3 - y^2x \right] \bigg|_{x=0}^{x=3} \, dy \]

\[ = \int_{y=0}^{y=3} (63 - 3y^2) \, dy = 63y - y^3 \bigg|_{y=0}^{y=3} = 162. \]

Exercise \( \PageIndex{5} \)

Find the volume of the solid bounded above by the graph of \( f(x,y) = xy \sin(x^2y) \) and below by the \( xy \)-plane on the rectangular region \( R = [0,1] \times [0,\pi] \).

**Hint**

Graph the function, set up the integral, and use an iterated integral.

**Answer**

\( \frac{\pi}{2} \)

Recall that we defined the average value of a function of one variable on an interval \([a,b] \) as

\[ f_{\text{ave}} = \frac{\int_a^b f(x) \, dx}{b-a} \]

Similarly, we can define the average value of a function of two variables over a region \( R \). The main difference is that we divide by an area instead of the width of an interval.

**Definition: Average Value of a Function**

The **average value of a function** of two variables over a region \( R \) is

\[ \bar{f} = \frac{1}{\text{Area of } R} \iint_R f(x,y) \, dx \, dy \]

In the next example we find the average value of a function over a rectangular region. This is a good example of obtaining useful information for an integration by making individual measurements over a grid, instead of trying to find an algebraic expression for a function.

**Example \( \PageIndex{10} \): Calculating Average Storm Rainfall**

The weather map in Figure \( \PageIndex{9} \) shows an unusually moist storm system associated with the remnants of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of the Midwest on September 22–23, 2010. The area of rainfall measured 300 miles east to west and 250 miles north to south. Estimate the average rainfall over the entire area in those two days.
Figure (PageIndex{9}): Effects of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of southwest Wisconsin, southern Minnesota, and southeast South Dakota over a span of 300 miles east to west and 250 miles north to south.

Solution

Place the origin at the southwest corner of the map so that all the values can be considered as being in the first quadrant and hence all are positive. Now divide the entire map into six rectangles (m = 2) and (n = 3), as shown in Figure (PageIndex{9}). Assume \(f(x,y)\) denotes the storm rainfall in inches at a point approximately \(x\) miles to the east of the origin and \(y\) miles to the north of the origin. Let \(R\) represent the entire area of \((250 \times 300 = 75000)\) square miles. Then the area of each subrectangle is

\[
\Delta A = \frac{1}{6} (75000) = 12500.
\]

Assume \((x_{ij}*,y_{ij}*)\) are approximately the midpoints of each subrectangle \(R_{ij}\). Note the color-coded region at each of these points, and estimate the rainfall. The rainfall at each of these points can be estimated as:

- At \((x_{11}, y_{11})\), the rainfall is 0.08.
- At \((x_{12}, y_{12})\), the rainfall is 0.08.
- At \((x_{13}, y_{13})\), the rainfall is 0.01.
- At \((x_{21}, y_{21})\), the rainfall is 1.70.
- At \((x_{22}, y_{22})\), the rainfall is 1.74.
- At \((x_{23}, y_{23})\), the rainfall is 3.00.

Figure (PageIndex{10}): Storm rainfall with rectangular axes and showing the midpoints of each subrectangle.
According to our definition, the average storm rainfall in the entire area during those two days was

\[
\begin{align*}
f_{\text{ave}} &= \frac{1}{\text{Area } R} \int \int_R f(x,y) \, dx \, dy = \frac{1}{75000} \int \int_R f(x,y) \, dx \, dy \\
&\approx \frac{1}{75000} \sum_{i=1}^3 \sum_{j=1}^2 f(x_{ij}^*, y_{ij}^*) \Delta A \\
&= \frac{1}{75000} \left[ f(x_{11}^*, y_{11}^*) \Delta A + f(x_{12}^*, y_{12}^*) \Delta A + f(x_{13}^*, y_{13}^*) \Delta A + f(x_{21}^*, y_{21}^*) \Delta A + f(x_{22}^*, y_{22}^*) \Delta A + f(x_{23}^*, y_{23}^*) \Delta A \right] \\
&\approx \frac{1}{75000} \left[ 0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00 \right] \Delta A \\
&= \frac{1}{6} \left[ 0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00 \right] \\
&\approx 1.10 \, \text{in}.
\end{align*}
\]

During September 22–23, 2010 this area had an average storm rainfall of approximately 1.10 inches.

Exercise \(\PageIndex{6}\)

A contour map is shown for a function \(f(x,y)\) on the rectangle \(R = [-3,6] \times [-1,4]\).

![Contour Map](image)

a. Use the midpoint rule with \(m = 3\) and \(n = 2\) to estimate the value of \(\iiint_R f(x,y) \, dA\).

b. Estimate the average value of the function \(f(x,y)\).

**Hint**

Divide the region into six rectangles, and use the contour lines to estimate the values for \(f(x,y)\).

**Answer**

Answers to both parts a. and b. may vary.

**Key Concepts**

- We can use a double Riemann sum to approximate the volume of a solid bounded above by a function of two variables over a rectangular region. By taking the limit, this becomes a double integral representing the volume of the solid.
Properties of double integral are useful to simplify computation and find bounds on their values.

We can use Fubini’s theorem to write and evaluate a double integral as an iterated integral.

Double integrals are used to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region.

---

**Key Equations**

- \[
  \int_{R} f(x,y) \, dA = \lim_{(m,n) \to (\infty,\infty)} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A
  \]

- \[
  \int_{a}^{b} \int_{c}^{d} f(x,y) \, dx \, dy = \int_{a}^{b} \left[ \int_{c}^{d} f(x,y) \, dy \right] dx
  \]
  or
  \[
  \int_{c}^{d} \int_{a}^{b} f(x,y) \, dx \, dy = \int_{c}^{d} \left[ \int_{a}^{b} f(x,y) \, dx \right] dy
  \]

- \[
  f_{ave} = \frac{1}{\text{Area of } R} \int_{R} f(x,y) \, dx \, dy
  \]

---

**Glossary**

**Double integral**

of the function \( f(x,y) \) over the region \((R)\) in the \((xy)\)-plane is defined as the limit of a double Riemann sum,

\[
\int_{R} f(x,y) \, dA = \lim_{(m,n) \to (\infty,\infty)} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A
\]

**Double Riemann sum**

of the function \( f(x,y) \) over a rectangular region \((R)\) is

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A
\]

where \((R)\) is divided into smaller subrectangles \((R_{ij})\) and \((x_{ij}^*, y_{ij}^*)\) is an arbitrary point in \((R_{ij})\).

**Fubini’s theorem**

if \( f(x,y) \) is a function of two variables that is continuous over a rectangular region \((R = \{ (x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, \, c \leq y \leq d \}\))

then the double integral of \( f \) over the region equals an iterated integral,

\[
\int_{R} f(x,y) \, dA = \int_{a}^{b} \left[ \int_{c}^{d} f(x,y) \, dy \right] dx
\]

**Iterated integral**

for a function \( f(x,y) \) over the region \((R)\) is

a. \[
\int_{a}^{b} \int_{c}^{d} f(x,y) \, dx \, dy = \int_{a}^{b} \left[ \int_{c}^{d} f(x,y) \, dx \right] dy
\]

b. \[
\int_{c}^{d} \int_{a}^{b} f(x,y) \, dx \, dy = \int_{c}^{d} \left[ \int_{a}^{b} f(x,y) \, dx \right] dy
\]

where \((a,b,c,d)\), and \((dl)\) are any real numbers and \((R = [a,b] \times [c,d])\)

---

**Contributors and Attributions**

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at [http://cnx.org](http://cnx.org).