15.8: Triple Integrals in Spherical Coordinates

Learning Objectives

- Evaluate a triple integral by changing to cylindrical coordinates.
- Evaluate a triple integral by changing to spherical coordinates.

Earlier in this chapter we showed how to convert a double integral in rectangular coordinates into a double integral in polar coordinates in order to deal more conveniently with problems involving circular symmetry. A similar situation occurs with triple integrals, but here we need to distinguish between cylindrical symmetry and spherical symmetry. In this section we convert triple integrals in rectangular coordinates into a triple integral in either cylindrical or spherical coordinates.

Also recall the chapter prelude, which showed the opera house l’Hemisphèric in Valencia, Spain. It has four sections with one of the sections being a theater in a five-story-high sphere (ball) under an oval roof as long as a football field. Inside is an IMAX screen that changes the sphere into a planetarium with a sky full of \(9000\) twinkling stars. Using triple integrals in spherical coordinates, we can find the volumes of different geometric shapes like these.

Review of Cylindrical Coordinates

As we have seen earlier, in two-dimensional space \(\mathbb{R}^2\) a point with rectangular coordinates \((x,y)\) can be identified with \((r,\theta)\) in polar coordinates and vice versa, where \(x = r \cos \theta\), \(y = r \sin \theta\), \(r^2 = x^2 + y^2\) and \(\tan \theta = \frac{y}{x}\) are the relationships between the variables.

In three-dimensional space \(\mathbb{R}^3\) a point with rectangular coordinates \((x,y,z)\) can be identified with cylindrical coordinates \((r,\theta,z)\) and vice versa. We can use these same conversion relationships, adding \(z\) as the vertical distance to the point from the \((xy)\)-plane as shown in \(\PageIndex{1}\).
Figure 1: Cylindrical coordinates are similar to polar coordinates with a vertical coordinate added.

To convert from rectangular to cylindrical coordinates, we use the conversion:

- \( x = r \cos \theta \)
- \( y = r \sin \theta \)
- \( z = z \)

To convert from cylindrical to rectangular coordinates, we use:

- \( r^2 = x^2 + y^2 \) and \( \theta = \tan^{-1} \left( \frac{y}{x} \right) \)
- \( z = z \)

Note that the \( z \)-coordinate remains the same in both cases.

In the two-dimensional plane with a rectangular coordinate system, when we say \( x = k \) (constant) we mean an unbounded vertical line parallel to the \( y \)-axis and when \( y = l \) (constant) we mean an unbounded horizontal line parallel to the \( x \)-axis. With the polar coordinate system, when we say \( r = c \) (constant), we mean a circle of radius \( c \) units and when \( \theta = \alpha \) (constant) we mean an infinite ray making an angle \( \alpha \) with the positive \( x \)-axis.

Similarly, in three-dimensional space with rectangular coordinates \( (x, y, z) \) the equations \( x = k \), \( y = l \) and \( z = m \) where \( k \), \( l \) and \( m \) are constants, represent unbounded planes parallel to the \( yz \)-plane, \( xz \)-plane and \( xy \)-plane, respectively. With cylindrical coordinates \( (r, \theta, z) \), by \( r = c \), \( \theta = \alpha \), and \( z = m \), where \( c \), \( \alpha \), and \( m \) are constants, we mean an unbounded vertical cylinder with the \( z \)-axis as its radial axis; a plane making a constant angle \( \alpha \) with the \( xy \)-plane; and an unbounded horizontal plane parallel to the \( xy \)-plane, respectively. This means that the circular cylinder \( x^2 + y^2 = c^2 \) in rectangular coordinates can be...
represented simply as \((r = c)\) in cylindrical coordinates. (Refer to Cylindrical and Spherical Coordinates for more review.)

**Integration in Cylindrical Coordinates**

Triple integrals can often be more readily evaluated by using cylindrical coordinates instead of rectangular coordinates. Some common equations of surfaces in rectangular coordinates along with corresponding equations in cylindrical coordinates are listed in Table \((\PageIndex{1})\). These equations will become handy as we proceed with solving problems using triple integrals.

<table>
<thead>
<tr>
<th>Cylindrical</th>
<th>Circular cone</th>
<th>Sphere</th>
<th>Paraboloid</th>
</tr>
</thead>
<tbody>
<tr>
<td>((r = c))</td>
<td>((z = cr))</td>
<td>((r^2 + z^2 = c^2))</td>
<td>((z = cr^2))</td>
</tr>
<tr>
<td>((x^2 + y^2 = c^2))</td>
<td>((z^2 = c^2 (x^2 + y^2)))</td>
<td>((x^2 + y^2 + z^2 = c^2))</td>
<td>((z = c(x^2 + y^2)))</td>
</tr>
</tbody>
</table>

As before, we start with the simplest bounded region \((B)\) in \((\mathbb{R}^3)\) to describe in cylindrical coordinates, in the form of a cylindrical box, \((B = \{(r, \theta, z) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta, c \leq z \leq d\})\) (Figure \((\PageIndex{2})\)). Suppose we divide each interval into \((l, m, n)\) subdivisions such that \((\Delta r = \frac{b - a}{l}, \Delta \theta = \frac{\beta - \alpha}{m}, \Delta z = \frac{d - c}{n})\). Then we can state the following definition for a triple integral in cylindrical coordinates.

\[
B_{ijk} = [r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j] \times [z_{k-1}, z_k]
\]

As before, we start with the simplest bounded region \((B)\) in \((\mathbb{R}^3)\) to describe in cylindrical coordinates, in the form of a cylindrical box, \((B = \{(r, \theta, z) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta, c \leq z \leq d\})\) (Figure \((\PageIndex{2})\)). Suppose we divide each interval into \((l, m, n)\) subdivisions such that \((\Delta r = \frac{b - a}{l}, \Delta \theta = \frac{\beta - \alpha}{m}, \Delta z = \frac{d - c}{n})\). Then we can state the following definition for a triple integral in cylindrical coordinates.

As before, we start with the simplest bounded region \((B)\) in \((\mathbb{R}^3)\) to describe in cylindrical coordinates, in the form of a cylindrical box, \((B = \{(r, \theta, z) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta, c \leq z \leq d\})\) (Figure \((\PageIndex{2})\)). Suppose we divide each interval into \((l, m, n)\) subdivisions such that \((\Delta r = \frac{b - a}{l}, \Delta \theta = \frac{\beta - \alpha}{m}, \Delta z = \frac{d - c}{n})\). Then we can state the following definition for a triple integral in cylindrical coordinates.

DEFINITION: triple integral in cylindrical coordinates

Consider the cylindrical box (expressed in cylindrical coordinates)
\[ B = \{(r, \theta, z) | a \leq r \leq b, \alpha \leq \theta \leq \beta, c \leq z \leq d\} \]

If the function \( f(r, \theta, z) \) is continuous on \( B \) and if \((r_{ijk}^*, \theta_{ijk}^*, z_{ijk}^*)\) is any sample point in the cylindrical subbox \( B_{ijk} = [r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j] \times [z_{k-1}, z_k] \) (Figure \( \PageIndex{2} \)), then we can define the triple integral in cylindrical coordinates as the limit of a triple Riemann sum, provided the following limit exists:

\[
\lim_{l,m,n \to \infty} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(r_{ijk}^*, \theta_{ijk}^*, z_{ijk}^*) \Delta r \Delta \theta \Delta z.
\]

Note that if \( g(x,y,z) \) is the function in rectangular coordinates and the box \( B \) is expressed in rectangular coordinates, then the triple integral

\[
\iiint_B g(x,y,z) dV
\]

is equal to the triple integral

\[
\iiint_B g(r, \cos \theta, \sin \theta, \theta, \theta, \theta, z) r \, dr \, d\theta \, dz
\]

and we have

\[
\iiint_B g(x,y,z) dV = \iiint_B g(r, \cos \theta, \sin \theta, \theta, \theta, \theta, z) r \, dr \, d\theta \, dz = \iiint_B f(r, \theta, z) r \, dr \, d\theta \, dz
\]

As mentioned in the preceding section, all the properties of a double integral work well in triple integrals, whether in rectangular coordinates or cylindrical coordinates. They also hold for iterated integrals. To reiterate, in cylindrical coordinates, Fubini’s theorem takes the following form:

Theorem: Fubini’s Theorem in Cylindrical Coordinates

Suppose that \( g(x,y,z) \) is continuous on a rectangular box \( B \) which when described in cylindrical coordinates looks like \( B = \{(r, \theta, z) | a \leq r \leq b, \alpha \leq \theta \leq \beta, c \leq z \leq d\} \).

Then \( g(x,y,z) = g(r, \cos \theta, \sin \theta, \theta, \theta, \theta, z) = f(r, \theta, z) \) and

\[
\iiint_B g(x,y,z) dV = \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} f(r, \theta, z) r \, dr \, d\theta \, dz.
\]

The iterated integral may be replaced equivalently by any one of the other five iterated integrals obtained by integrating with respect to the three variables in other orders.

Cylindrical coordinate systems work well for solids that are symmetric around an axis, such as cylinders and cones. Let us look at some examples before we define the triple integral in cylindrical coordinates on general cylindrical regions.

Example \( \PageIndex{1} \): Evaluating a Triple Integral over a Cylindrical Box

Evaluate the triple integral

\[
\iiint_B (zr \, \sin \, \theta) r \, dr \, d\theta \, dz
\]

where the cylindrical box \( B \) is \( B = \{(r, \theta, z) | 0 \leq r \leq 2, 0 \leq \theta \leq \theta, 0 \leq z \leq 4\} \).

**Solution**

As stated in Fubini’s theorem, we can write the triple integral as the iterated integral
\[
\iiint_B (zr \, \sin \, \theta) r \, dr \, d\theta \, dz = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2} \int_{z=0}^{z=4} (zr \, \sin \, \theta) r \, dz \, dr \, d\theta.
\]

The evaluation of the iterated integral is straightforward. Each variable in the integral is independent of the others, so we can integrate each variable separately and multiply the results together. This makes the computation much easier:

\[
\int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2} \int_{z=0}^{z=4} (zr \, \sin \, \theta) r \, dz \, dr \, d\theta = \\
\left(\int_0^{\pi/2} \sin \, \theta \, d\theta \right) \left( \int_0^2 r^2 \, dr \right) \left( \int_0^4 z \, dz \right) = \\
\left(\left. -\cos \theta \right|_0^{\pi/2} \right) \left(\left. \frac{r^3}{3} \right|_0^2 \right) \left( \left. \frac{z^2}{2} \right|_0^4 \right) = \\
\frac{64}{3}.
\]

Exercise \(\PageIndex{1}\):

Evaluate the triple integral \[
\int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=4} rz \, \sin \, \theta r \, dz \, dr \, d\theta.
\]

**Hint**

Follow the same steps as in the previous example.

**Answer**

\((8)\)

If the cylindrical region over which we have to integrate is a general solid, we look at the projections onto the coordinate planes. Hence the triple integral of a continuous function \(f(r, \theta, z)\) over a general solid region \(E = \{(r, \theta, z) | (r, \theta) \in D, u_1 (r, \theta) \leq z \leq u_2 (r, \theta)\}\) in \(\mathbb{R}^3\) where \(D\) is the projection of \(E\) onto the \((r\theta)\)-plane, is

\[
\iiint_E f(r, \theta, z) r \, dr \, d\theta \, dz = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} \int_{z=u_1(r,\theta)}^{z=u_2(r,\theta)} f(r, \theta, z) r \, dz \, dr \, d\theta.
\]

Similar formulas exist for projections onto the other coordinate planes. We can use polar coordinates in those planes if necessary.

**Example \(\PageIndex{2}\): Setting up a Triple Integral in Cylindrical Coordinates over a General Region**

Consider the region \(E\) inside the right circular cylinder with equation \(r = 2 \ \sin \ \theta\), bounded below by the \((r\theta)\)-plane and bounded above by the sphere with radius \(4\) centered at the origin (Figure 15.5.3). Set up a triple integral over this region with a function \(f(r, \theta, z)\) in cylindrical coordinates.
Solution

First, identify that the equation for the sphere is \(r^2 + z^2 = 16\). We can see that the limits for \(z\) are from \(0\) to \(z = \sqrt{16 - r^2}\). Then the limits for \(r\) are from \(0\) to \(r = 2 \sin \theta\). Finally, the limits for \(\theta\) are from \(0\) to \(\pi\). Hence the region is \(E = \{(r,\theta, z)|0 \leq \theta \leq \pi, \, 0 \leq r \leq 2 \sin \theta, \, 0 \leq z \leq \sqrt{16 - r^2} \}\). Therefore, the triple integral is

\[
\iiint_E f(r,\theta, z) r \, dz \, dr \, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2 \sin \theta} \int_{z=0}^{z=\sqrt{16-r^2}} f(r,\theta,z) r \, dz \, dr \, d\theta.
\]

Exercise \(\PageIndex{2}\):

Consider the region inside the right circular cylinder with equation \(r=2 \sin \theta\) bounded below by the \((r, \theta)\)-plane and bounded above by \(z = 4 - y\). Set up a triple integral with a function \((f(r,\theta,z))\) in cylindrical coordinates.

Hint

Analyze the region, and draw a sketch.

Answer

\[
\iiint_E f(r,\theta, z) r \, dz \, dr \, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2 \sin \theta} \int_{z=0}^{z=4-r \sin \theta} f(r,\theta,z) r \, dz \, dr \, d\theta.
\]

Example \(\PageIndex{3}\): Setting up a Triple Integral in Two Ways

Let \(E\) be the region bounded below by the cone \(z = \sqrt{x^2 + y^2}\) and above by the paraboloid \(z = 2 - x^2 - y^2\), (Figure 15.5.4). Set up a triple integral in cylindrical coordinates to find the volume of the region, using the
following orders of integration:

a. \((dz \, dr \, d\theta)\)

b. \((dr \, dz \, d\theta)\)

Figure \(\PageIndex{4}\): Setting up a triple integral in cylindrical coordinates over a conical region.

Solution

a. The cone is of radius 1 where it meets the paraboloid. Since \(z = 2 - x^2 - y^2 = 2 - r^2\) and \(z = \sqrt{x^2 + y^2} = r^2\) (assuming \(r\) is nonnegative), we have \(2 - r^2 = r\). Solving, we have \(r^2 + r - 2 = (r + 2)(r - 1) = 0\). Since \(r \geq 0\), we have \(r = 1\). Therefore \(z = 1\). So the intersection of these two surfaces is a circle of radius \(1\) in the plane \(z = 1\). The cone is the lower bound for \(z\) and the paraboloid is the upper bound. The projection of the region onto the \((xy)\)-plane is the circle of radius \(1\) centered at the origin.

Thus, we can describe the region as \(E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi, \, 0 \leq r \leq 1, \, r \leq z \leq 2 - r^2 \}\).

Hence the integral for the volume is

\[V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} r \, dz \, dr \, d\theta.\]

b. We can also write the cone surface as \(r = z\) and the paraboloid as \(r^2 = 2 - z\). The lower bound for \(r\) is zero, but the upper bound is sometimes the cone and the other times it is the paraboloid. The plane \(z = 1\) divides the region into two regions. Then the region can be described as \(E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi, \, 0 \leq z \leq 1, \, 0 \leq r \leq z \}\) \cup \(\{(r, \theta, z) | 0 \leq \theta \leq 2\pi, \, 1 \leq z \leq 2, \, 0 \leq r \leq \sqrt{2 - z} \}\).

Now the integral for the volume becomes

\[V = \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} \int_{r=0}^{r=z} r \, dr \, dz \, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=2} \int_{r=0}^{r=\sqrt{2-z}} r \, dr \, dz \, d\theta.\]

Exercise \(\PageIndex{3}\):
Redo the previous example with the order of integration \( d\theta \, dz \, dr \).

**Hint**

Note that \( \theta \) is independent of \( r \) and \( z \).

**Answer**

\[
E = \{(r,\theta,z)|0 \leq \theta \leq 2\pi, \, 0 \leq z \leq 1, \, 0 \leq r \leq 2 - z^2\}\) and \[V = \int_{r=0}^{r=1} \int_{z=0}^{z=2 - r^2} \int_{\theta=0}^{\theta=2\pi} r \, d\theta \, dz \, dr.\]

Example \(\PageIndex{4}\): Finding a Volume with Triple Integrals in Two Ways

Let \( E \) be the region bounded below by the \((r,\theta)-\)plane, above by the sphere \((x^2 + y^2 + z^2 = 4)\), and on the sides by the cylinder \((x^2 + y^2 = 1)\) (Figure 15.5.5). Set up a triple integral in cylindrical coordinates to find the volume of the region using the following orders of integration, and in each case find the volume and check that the answers are the same:

a. \(dz \, dr \, d\theta\)

b. \(dr \, dz \, d\theta\).

**Solution**

a. Note that the equation for the sphere is

\[
x^2 + y^2 + z^2 = 4 \text{ (or)} \, r^2 + z^2 = 4
\]

and the equation for the cylinder is
Thus, we have for the region \( \Psi(E) \)

\[
\Psi(E) = \{(r, \theta, z) | 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi, \ 0 \leq z \leq \sqrt{4 - r^2}\}
\]

Hence the integral for the volume is

\[
\begin{align}
V(E) &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{4-r^2}} r \, dz \, dr \, d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left[ rz \right]_{z=0}^{z=\sqrt{4-r^2}} dr \, d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left(r\sqrt{4 - r^2}\right) dr \, d\theta \\
&= \int_0^{2\pi} \left(\frac{8}{3} - \sqrt{3}\right) d\theta \\
&= 2\pi \left(\frac{8}{3} - \sqrt{3}\right) \, \text{cubic units.}
\end{align}
\]

b. Since the sphere is \((x^2 + y^2 + z^2 = 4)\), which is \((r^2 + z^2 = 4)\), and the cylinder is \((x^2 + y^2 = 1)\), which is \((r^2 = 1)\), we have \((1 + z^2 = 4)\), that is, \((z^2 = 3)\). Thus we have two regions, since the sphere and the cylinder intersect at \((1,\sqrt{3})\) in the \((rz)\)-plane

\[
\Psi(E_1) = \{(r, \theta, z) | 0 \leq r \leq \sqrt{4 - r^2}, \ \sqrt{3} \leq z \leq 2, \ 0 \leq \theta \leq 2\pi\}
\]

\[
\Psi(E_2) = \{(r, \theta, z) | 0 \leq r \leq 1, \ 0 \leq z \leq \sqrt{3}, \ 0 \leq \theta \leq 2\pi\}
\]

Hence the integral for the volume is

\[
\begin{align}
V(E) &= \int_{\theta=0}^{\theta=2\pi} \int_{z=\sqrt{3}}^{z=2} \int_{r=0}^{r=\sqrt{4-r^2}} r \, dr \, dz \, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=\sqrt{3}} \int_{r=0}^{r=1} r \, dr \, dz \, d\theta \\
&= \sqrt{3}\pi + \left(\frac{16}{3} - 3\sqrt{3}\right)\pi \\
&= 2\pi \left(\frac{8}{3} - \sqrt{3}\right) \, \text{cubic units.}
\end{align}
\]

Exercise \(\PageIndex{4}\)

Redo the previous example with the order of integration \((d\theta, \, dz, \, dr)\).

**Hint**

A figure can be helpful. Note that \((\theta)\) is independent of \((r)\) and \((z)\).

**Answer**

\[
\begin{align}
\Psi(E_2) &= \{(r, \theta, z) | 0 \leq \theta \leq 2\pi, \ 0 \leq r \leq 1, \ r \leq z \leq \sqrt{4 - r^2}\} \) and \\
V &= \int_{r=0}^{r=1} \int_{z=r}^{z=\sqrt{4-r^2}} \int_{\theta=0}^{\theta=2\pi} \, d\theta \, dz \, dr \\
&= \text{cubic units.}
\end{align}
\]

**Review of Spherical Coordinates**

In three-dimensional space \((\mathbb{R}^3)\) in the spherical coordinate system, we specify a point \((P)\) by its distance \((\rho)\) from the origin, the polar angle \((\theta)\) from the positive \((x)\)-axis (same as in the cylindrical coordinate system), and the angle \((\varphi)\) from the positive \((x)\)-axis and the line \((OP)\) (Figure \(\PageIndex{6}\)). Note that \((\rho > 0)\) and \((0 \leq \varphi \leq \pi)\). (Refer to Cylindrical and Spherical Coordinates for a review.) Spherical coordinates are useful for triple integrals over regions that are symmetric with respect to the origin.
Figure \(\PageIndex{6}\): The spherical coordinate system locates points with two angles and a distance from the origin.

Recall the relationships that connect rectangular coordinates with spherical coordinates.

From spherical coordinates to rectangular coordinates:

\[x = \rho \, \sin \, \varphi \, \cos \theta, \, y = \rho \, \sin \, \varphi \, \sin \, \theta, \, \text{and} \, \, z = \rho \, \cos \, \varphi.\]

From rectangular coordinates to spherical coordinates:

\[\rho^2 = x^2 + y^2 + z^2, \, \tan \, \theta = \frac{y}{x}, \, \varphi = \arccos \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right).\]

Other relationships that are important to know for conversions are

- \(r = \rho \, \sin \, \varphi\)
- \((\theta = \theta)\) These equations are used to convert from spherical coordinates to cylindrical coordinates.
- \((z = \rho \, \cos \, \varphi)\)

and

- \((\rho = \sqrt{r^2 + z^2})\)
- \((\theta = \theta)\) These equations are used to convert from cylindrical coordinates to spherical coordinates.
- \((\varphi = \arccos \left( \frac{z}{\sqrt{r^2 + z^2}} \right))\)

\(\PageIndex{7}\) shows a few solid regions that are convenient to express in spherical coordinates.
Integration in Spherical Coordinates

We now establish a triple integral in the spherical coordinate system, as we did before in the cylindrical coordinate system. Let the function \( f(\rho, \theta, \varphi) \) be continuous in a bounded spherical box, \( B = \{ (\rho, \theta, \varphi) \mid a \leq \rho \leq b, \, \alpha \leq \theta \leq \beta, \, \gamma \leq \varphi \leq \psi \} \). We then divide each interval into \((l,m,n)\) and \((n)\) subdivisions such that \( \Delta \rho = \frac{b - a}{l}, \, \Delta \theta = \frac{\beta - \alpha}{m}, \, \Delta \varphi = \frac{\psi - \gamma}{n} \). Now we can illustrate the following theorem for triple integrals in spherical coordinates with \( (\rho_{ijk}^*, \theta_{ijk}^*, \varphi_{ijk}^*) \) being any sample point in the spherical subbox \( B_{ijk} \). For the volume element of the subbox \( \Delta V \) in spherical coordinates, we have \( \Delta V = (\Delta \rho, \, \rho \Delta \varphi, \, \rho \sin \varphi \, \Delta \theta) \), as shown in the following figure.
Definition: triple integral in spherical coordinates

The \textbf{triple integral in spherical coordinates} is the limit of a triple Riemann sum,

$$
\lim_{l,m,n \rightarrow \infty} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f (\rho_{ijk}^*, \theta_{ijk}^*, \varphi_{ijk}^*) (\rho_{ijk}^*)^2 \sin \varphi \Delta \rho \Delta \theta \Delta \varphi
$$

provided the limit exists.

As with the other multiple integrals we have examined, all the properties work similarly for a triple integral in the spherical coordinate system, and so do the iterated integrals. Fubini’s theorem takes the following form.

Theorem: Fubini’s Theorem for Spherical Coordinates

If \(f(\rho,\theta, \varphi)\) is continuous on a spherical solid box \(B = [a,b] \times [\alpha,\beta] \times [\gamma , \psi]\), then

$$
\iiint_B f(\rho,\theta,\varphi) \, \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta =
\int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho,\theta,\varphi) \, \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.
$$

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

As stated before, spherical coordinate systems work well for solids that are symmetric around a point, such as spheres and cones. Let us look at some examples before we consider triple integrals in spherical coordinates on general spherical regions.

Example \(\PageIndex{5}\): Evaluating a Triple Integral in Spherical Coordinates

Evaluate the iterated triple integral

$$
\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.
$$

Solution

As before, in this case the variables in the iterated integral are actually independent of each other and hence we can integrate each piece and multiply:

$$
\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = (2\pi) \cdot (1) \cdot \left(\frac{1}{3}\right) = \frac{2\pi}{3}
$$

The concept of triple integration in spherical coordinates can be extended to integration over a general solid, using the projections onto the coordinate planes. Note that \((dV)\) and \((dA)\) mean the increments in volume and area, respectively. The variables \((V)\) and \((A)\) are used as the variables for integration to express the integrals.

The triple integral of a continuous function \(f(\rho,\theta,\varphi)\) over a general solid region

$$
[E = \{(\rho,\theta,\varphi) | (\rho,\theta) \in D, u_1 (\rho, \theta) \leq \varphi \leq u_2 (\rho, \theta)\}]
$$

in \((\text{mathbb}{R}^3)\), where \((D)\) is the projection of \((E)\) onto the \((\rho \theta)\)-plane, is
In particular, if \( D = \{(\rho, \theta)| g_1(\theta) \leq \rho \leq g_2(\theta), \alpha \leq \theta \leq \beta\} \), we have

\[
\iiint_E f(\rho, \theta, \varphi) \, dV = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} \int_{u_1(\rho, \theta)}^{u_2(\rho, \theta)} f(\rho, \theta, \varphi) \rho^2 \sin \varphi \, d\varphi \, d\rho \, d\theta.
\]

Similar formulas occur for projections onto the other coordinate planes.

Example \( \PageIndex{6} \): Setting up a Triple Integral in Spherical Coordinates

Set up an integral for the volume of the region bounded by the cone \( z = \sqrt{3(x^2 + y^2)} \) and the hemisphere \( z = \sqrt{4 - x^2 - y^2} \) (see the figure below).

![Figure \( \PageIndex{9} \): A region bounded below by a cone and above by a hemisphere.](image)

**Solution**

Using the conversion formulas from rectangular coordinates to spherical coordinates, we have:

For the cone: \( z = \sqrt{3(x^2 + y^2)} \) or \( \rho \cos \varphi = \sqrt{3} \rho \sin \varphi \) or \( \tan \varphi = \frac{1}{\sqrt{3}} \) or \( \varphi = \frac{\pi}{6} \).

For the sphere: \( z = \sqrt{4 - x^2 - y^2} \) or \( z^2 + x^2 + y^2 = 4 \) or \( \rho^2 = 4 \) or \( \rho = 2 \).

Thus, the triple integral for the volume is

\[
V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi+\pi/6} \int_{\rho=0}^{\rho=2} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.
\]

Exercise \( \PageIndex{5} \)

Set up a triple integral for the volume of the solid region bounded above by the sphere \( \rho = 2 \) and bounded below
by the cone $\varphi = \pi/3$.

**Hint**

Follow the steps of the previous example.

**Answer**

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/3} \int_{\rho=0}^{\rho=2} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$

**Example $\PageIndex{7}$: Interchanging Order of Integration in Spherical Coordinates**

Let $E$ be the region bounded below by the cone $(z = \sqrt{x^2 + y^2})$ and above by the sphere $(z = x^2 + y^2 + z^2)$ (Figure 15.5.10). Set up a triple integral in spherical coordinates and find the volume of the region using the following orders of integration:

a. $d\rho \, d\varphi \, d\theta$

b. $d\varphi \, d\rho \, d\theta$

**Solution**

a. Use the conversion formulas to write the equations of the sphere and cone in spherical coordinates.

For the sphere:

$$\begin{align*} x^2 + y^2 + z^2 &= z \\
\rho^2 &= \rho \cos \varphi \\
\rho &= \cos \varphi. \end{align*}$$

For the cone:

$$\begin{align*} z &= \sqrt{x^2 + y^2} \\
\rho \cos \varphi &= \sqrt{\rho^2 \sin^2 \varphi \cos^2 \phi + \sin^2 \phi} \\
\rho \cos \varphi &= \rho \sin \varphi \\
\cos \varphi &= \sin \phi. \end{align*}$$

Figure $\PageIndex{10}$: A region bounded below by a cone and above by a sphere.
\[ \varphi = \sin \varphi \quad \varphi = \pi/4. \]

Hence the integral for the volume of the solid region \( E \) becomes

\[ V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/4} \int_{\rho=0}^{\rho=\cos \varphi} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta. \]

b. Consider the \((\varphi,\rho)\)-plane. Note that the ranges for \( \varphi \) and \( \rho \) (from part a.) are

\[ \begin{align*}
0 \leq \rho \leq \sqrt{2}/2 & \quad \text{and} \quad \sqrt{2} \leq \rho \leq 1 \\
0 \leq \varphi \leq \pi/4 & \quad 0 \leq \rho \leq \cos \varphi 
\end{align*} \]

The curve \((\rho = \cos \varphi)\) meets the line \((\varphi = \pi/4)\) at the point \((\pi/4, \sqrt{2}/2)\). Thus, to change the order of integration, we need to use two pieces:

\[ \begin{align*}
0 \leq \rho \leq \sqrt{2}/2, & \quad 0 \leq \varphi \leq \pi/4 \\
\sqrt{2}/2 \leq \rho \leq 1, & \quad 0 \leq \varphi \leq \cos^{-1} \rho
\end{align*} \]

Hence the integral for the volume of the solid region \( E \) becomes

\[ V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=\sqrt{2}/2} \int_{\varphi=0}^{\varphi=\pi/4} \rho^2 \sin \varphi \, d\varphi \, d\rho \, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\sqrt{2}/2}^{\rho=1} \int_{\varphi=0}^{\varphi=\cos^{-1} \rho} \rho^2 \sin \varphi \, d\varphi \, d\rho \, d\theta. \]

In each case, the integration results in \( V(E) = \frac{\pi}{8} \).

Before we end this section, we present a couple of examples that can illustrate the conversion from rectangular coordinates to cylindrical coordinates and from rectangular coordinates to spherical coordinates.

Example \( \PageIndex{8} \): Converting from Rectangular Coordinates to Cylindrical Coordinates

Convert the following integral into cylindrical coordinates:

\[ \int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy. \]

Solution

The ranges of the variables are

\[ \begin{align*}
-1 \leq y & \leq 1 \\
0 \leq x & \leq \sqrt{1-y^2} \\
x^2 + y^2 \leq z & \leq \sqrt{x^2 + y^2}
\end{align*} \]

The first two inequalities describe the right half of a circle \((1)\). Therefore, the ranges for \( (\theta) \) and \( (r) \) are

\[ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 1. \]

The limits of \((z)\) are \((r^2 \leq z \leq r)\), hence

\[ \begin{align*}
\int_{y=-1}^{y=1} & \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy = \\
\int_{\theta=-\pi/2}^{\theta=\pi/2} & \int_{r=0}^{r=\sqrt{1}} \int_{r}^{\cos \theta} r^2 (r \, \cos \theta) \, r \, \sin \theta \, dz \, dr \, d\theta.
\end{align*} \]

Example \( \PageIndex{9} \): Converting from Rectangular Coordinates to Spherical Coordinates

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Convert the following integral into spherical coordinates:

\[
\int_{y=0}^{y=3} \int_{x=0}^{x=\sqrt{9-y^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2) \, dz \, dx \, dy.
\]

**Solution**

The ranges of the variables are

\[
\begin{align}
0 \leq y \leq 3 \\
0 \leq x \leq \sqrt{9 - y^2} \\
\sqrt{x^2 + y^2} \leq z \leq \sqrt{18 - x^2 - y^2}.
\end{align}
\]

The first two ranges of variables describe a quarter disk in the first quadrant of the \((xy)\)-plane. Hence the range for \((\theta)\) is \(0 \leq \theta \leq \frac{\pi}{2}\).

The lower bound \((z = \sqrt{x^2 + y^2})\) is the upper half of a cone and the upper bound \((z = \sqrt{18 - x^2 - y^2})\) is the upper half of a sphere. Therefore, we have \(0 \leq \rho \leq \sqrt{18}\), which is \(0 \leq \rho \leq 3\sqrt{2}\).

For the ranges of \((\varphi)\) we need to find where the cone and the sphere intersect, so solve the equation

\[
\begin{align}
r^2 + z^2 &= 18 \\
\sqrt{x^2 + y^2}^2 + z^2 &= 18 \\
z^2 &= 9 \\
z &= 3.
\end{align}
\]

This gives

\[
\begin{align}
3\sqrt{2} \, \cos \, \varphi &= 3 \\
\cos \, \varphi &= \frac{1}{\sqrt{2}} \\
\varphi &= \frac{\pi}{4}.
\end{align}
\]

Putting this together, we obtain

\[
\int_{y=0}^{y=3} \int_{x=0}^{x=\sqrt{9-y^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2) \, dz \, dx \, dy = \int_{\varphi=0}^{\varphi=\pi/4} \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=3\sqrt{2}} \rho^4 \sin \, \varphi \, d\rho \, d\theta \, d\varphi.
\]

Exercise \((PageIndex(6))\):

Use rectangular, cylindrical, and spherical coordinates to set up triple integrals for finding the volume of the region inside the sphere \((x^2 + y^2 + z^2 = 4)\) but outside the cylinder \((x^2 + y^2 = 1)\).

**Answer: Rectangular**

\[
\int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz \, dy \, dx.
\]

**Answer: Cylindrical**

\[
\int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} \int_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}} r \, dz \, dr \, d\theta.
\]

**Answer: Spherical**

\[
\int_{\varphi=\pi/6}^{\varphi=5\pi/6} \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=\csc \, \varphi}^{\rho=2} \rho^2 \sin \, \varphi \, d\rho \, d\theta \, d\varphi.
\]

Now that we are familiar with the spherical coordinate system, let’s find the volume of some known geometric figures,
such as spheres and ellipsoids.

Example \(\PageIndex{10}\): Chapter Opener: Finding the Volume of l’Hemisphèric

Find the volume of the spherical planetarium in l’Hemisphèric in Valencia, Spain, which is five stories tall and has a radius of approximately \((50')\) ft, using the equation \((x^2 + y^2 + z^2 = r^2)\).

Figure \(\PageIndex{11}\): (credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

Solution

We calculate the volume of the ball in the first octant, where \((x \leq 0, y \leq 0, z \leq 0)\), using spherical coordinates, and then multiply the result by \(8\) for symmetry. Since we consider the region \(D\) as the first octant in the integral, the ranges of the variables are

\[
0 \leq \varphi \leq \frac{\pi}{2}, \quad 0 \leq \rho \leq r, \quad 0 \leq \theta \leq \frac{\pi}{2}.
\]

Therefore,

\[
\begin{align}
V &= \iiint_D dx \, dy \, dz = 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=r} \int_{\varphi=0}^{\varphi=\pi/2} \rho^2 \sin \theta \, d\varphi \, d\rho \, d\theta \\
&= 8 \int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=r} \rho^2 d\rho \int_{\theta=0}^{\theta=\pi/2} \sin \theta \, d\theta \\
&= 8 \left(\frac{\pi}{2}\right) \left(\frac{r^3}{3}\right) (1) \\
&= \frac{4}{3} \pi r^3.
\end{align}
\]

This exactly matches with what we knew. So for a sphere with a radius of approximately \((50')\) ft, the volume is \(\frac{4}{3} \pi (50)^3 \approx 523,600 \text{ ft}^3\).

For the next example we find the volume of an ellipsoid.

Example \(\PageIndex{11}\): Finding the Volume of an Ellipsoid
Find the volume of the ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \).

**Solution**

We again use symmetry and evaluate the volume of the ellipsoid using spherical coordinates. As before, we use the first octant \( (x \leq 0, y \leq 0) \), and \( z \leq 0 \) and then multiply the result by \( 8 \).

In this case the ranges of the variables are

\[
0 \leq \varphi \leq \frac{\pi}{2}, \quad 0 \leq \rho \leq 1, \quad 0 \leq \theta \leq \frac{\pi}{2}.
\]

Also, we need to change the rectangular to spherical coordinates in this way:

\[
x = a \rho \cos \varphi \sin \theta, \quad y = b \rho \sin \varphi \sin \theta, \quad z = c \rho \cos \theta.
\]

Then the volume of the ellipsoid becomes

\[
\begin{align*}
V = & \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=1} \int_{\varphi=0}^{\varphi=\pi/2} abc \rho^2 \sin \theta \, d\varphi \, d\rho \, d\theta \\
= & 8abc \left( \frac{\pi}{2} \right) \left( \frac{1}{3} \right) (1) \\
= & \frac{4}{3} \pi abc.
\end{align*}
\]

Example \( \PageIndex{12} \): Finding the Volume of the Space Inside an Ellipsoid and Outside a Sphere

Find the volume of the space inside the ellipsoid \( \frac{x^2}{75^2} + \frac{y^2}{80^2} + \frac{z^2}{90^2} = 1 \) and outside the sphere \( x^2 + y^2 + z^2 = 50^2 \).

**Solution**

This problem is directly related to the l’Hemisphèric structure. The volume of space inside the ellipsoid and outside the sphere might be useful to find the expense of heating or cooling that space. We can use the preceding two examples for the volume of the sphere and ellipsoid and then subtract.

First we find the volume of the ellipsoid using \( a = 75 \) ft, \( b = 80 \) ft, and \( c = 90 \) ft in the result from Example. Hence the volume of the ellipsoid is

\[
V_{\text{ellipsoid}} = \frac{4}{3} \pi (75)(80)(90) \approx 2,262,000 \text{ ft}^3.
\]

From Example, the volume of the sphere is

\[
V_{\text{sphere}} \approx 523,600 \text{ ft}^3.
\]

Therefore, the volume of the space inside the ellipsoid \( \frac{x^2}{75^2} + \frac{y^2}{80^2} + \frac{z^2}{90^2} = 1 \) and outside the sphere \( x^2 + y^2 + z^2 = 50^2 \) is approximately

\[
V_{\text{Hemispheric}} = V_{\text{ellipsoid}} - V_{\text{sphere}} = 1,738,400 \text{ ft}^3.
\]

**Student Project: Hot air balloons**

Hot air ballooning is a relaxing, peaceful pastime that many people enjoy. Many balloonist gatherings take place around the world, such as the Albuquerque International Balloon Fiesta. The Albuquerque event is the largest hot air balloon festival in the world, with over \( 500 \) balloons participating each year.
As the name implies, hot air balloons use hot air to generate lift. (Hot air is less dense than cooler air, so the balloon floats as long as the hot air stays hot.) The heat is generated by a propane burner suspended below the opening of the basket. Once the balloon takes off, the pilot controls the altitude of the balloon, either by using the burner to heat the air and ascend or by using a vent near the top of the balloon to release heated air and descend. The pilot has very little control over where the balloon goes, however—balloons are at the mercy of the winds. The uncertainty over where we will end up is one of the reasons balloonists are attracted to the sport.

In this project we use triple integrals to learn more about hot air balloons. We model the balloon in two pieces. The top of the balloon is modeled by a half sphere of radius 28 feet. The bottom of the balloon is modeled by a frustum of a cone (think of an ice cream cone with the pointy end cut off). The radius of the large end of the frustum is \(\frac{28}{2}\) feet and the radius of the small end of the frustum is \(\frac{28}{4}\) feet. A graph of our balloon model and a cross-sectional diagram showing the dimensions are shown in the following figure.
cone to model the bottom part of the balloon. (b) A cross section of the balloon showing its dimensions.

We first want to find the volume of the balloon. If we look at the top part and the bottom part of the balloon separately, we see that they are geometric solids with known volume formulas. However, it is still worthwhile to set up and evaluate the integrals we would need to find the volume. If we calculate the volume using integration, we can use the known volume formulas to check our answers. This will help ensure that we have the integrals set up correctly for the later, more complicated stages of the project.

1. Find the volume of the balloon in two ways.

a. Use triple integrals to calculate the volume. Consider each part of the balloon separately. (Consider using spherical coordinates for the top part and cylindrical coordinates for the bottom part.)

b. Verify the answer using the formulas for the volume of a sphere, \( V = \frac{4}{3}\pi r^3 \), and for the volume of a cone, \( V = \frac{1}{3}\pi r^2 h \).

In reality, calculating the temperature at a point inside the balloon is a tremendously complicated endeavor. In fact, an entire branch of physics (thermodynamics) is devoted to studying heat and temperature. For the purposes of this project, however, we are going to make some simplifying assumptions about how temperature varies from point to point within the balloon. Assume that just prior to liftoff, the temperature (in degrees Fahrenheit) of the air inside the balloon varies according to the function \( T_0(r,\theta,z) = \frac{z - r}{10} + 210 \).

2. What is the average temperature of the air in the balloon just prior to liftoff? (Again, look at each part of the balloon separately, and do not forget to convert the function into spherical coordinates when looking at the top part of the balloon.)

Now the pilot activates the burner for 10 seconds. This action affects the temperature in a 12-foot-wide column 20 feet high, directly above the burner. A cross section of the balloon depicting this column in shown in the following figure.

![Figure](\PageIndex{14}): Activating the burner heats the air in a 20-foot-high, 12-foot-wide
column directly above the burner.

Assume that after the pilot activates the burner for \(10\) seconds, the temperature of the air in the column described above increases according to the formula

\[H(r, \theta, z) = -2z - 48.\]

Then the temperature of the air in the column is given by \([T_1(r, \theta, z) = \frac{z - r}{10} + 210 + (-2z - 48),\]

while the temperature in the remainder of the balloon is still given by \([T_0(r, \theta, z) = \frac{z - r}{10} + 210.\]

3. Find the average temperature of the air in the balloon after the pilot has activated the burner for \(10\) seconds.

Key Concepts

- To evaluate a triple integral in cylindrical coordinates, use the iterated integral

\[
\int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} \int_{z=u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z) r \, dz \, dr \, d\theta.
\]

- To evaluate a triple integral in spherical coordinates, use the iterated integral

\[
\int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=g_1(\theta)}^{\rho=g_2(\theta)} \int_{\varphi=u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi) \rho^2 \sin \varphi \, d\varphi \, d\rho \, d\theta.
\]

Key Equations

- Triple integral in cylindrical coordinates

\[
\int_{\rho=a}^{\rho=b} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=\alpha}^{\rho=\beta} f(\rho, \theta, \varphi) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.
\]

- Triple integral in spherical coordinates

\[
\int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho, \theta, \varphi) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.
\]

Glossary

triple integral in cylindrical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

\[
\lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(r_{ijk}, \theta_{ijk}, \varphi_{ijk}) r_{ijk} \Delta r \Delta \theta \Delta z.
\]

triple integral in spherical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

\[
\lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\rho_{ijk}, \theta_{ijk}, \varphi_{ijk}) \rho_{ijk}^2 \sin \varphi \Delta \rho \Delta \theta \Delta \varphi.
\]
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