16.6: Parametric Surfaces and Their Areas

We have now seen many kinds of functions. When we talked about parametric curves, we defined them as functions from \(\mathbb{R}\) to \(\mathbb{R}^2\) (plane curves) or \(\mathbb{R}\) to \(\mathbb{R}^3\) (space curves). Because each of these has its domain \(\mathbb{R}\), they are one dimensional (you can only go forward or backward). In this section, we investigate how to parameterize two dimensional surfaces. Below is the definition.

**Definition: Parametric Surfaces**

A *parametric surface* is a function with domain \(\mathbb{R}^2\) and range \(\mathbb{R}^3\).

We typically use the variables \(u\) and \(v\) for the domain and \(x\), \(y\), and \(z\) for the range. We often use vector notation to exhibit parametric surfaces.

Example \(\PageIndex{1}\)

A sphere of radius 7 can be parameterized by

\[
\textbf{r}(u,v) = 7 \cos u \sin v \hat{i} + 7 \sin u \sin v \hat{j} + 7 \cos v \hat{k} \]

Notice that we have just used spherical coordinates with the radius held at 7.

We can use a computer to graph a parametric surface. Below is the graph of the surface

\[
\textbf{r}(u,v) = \sin u \hat{i} + \cos v \hat{j} + \exp (2u^{\frac{1}{3}} + 2v^{\frac{1}{3}}) \hat{k}
\]
Example (PageIndex{2})

Represent the surface

\[ z = e^x \cos(x-y) \]

parametrically.

Solution

The idea is similar to parametric curves. We just let \(x = u\) and \(y = v\), to get

\[
\mathbf{r}(u,v) = u \hat{i} + v \hat{j} + e^u \cos(u-v) \hat{k}.
\]

Example (PageIndex{3})

A surface is created by revolving the curve

\[ y = \cos x \]

about the x-axis. Find parametric equations for this surface.

Solution

For a fixed value of \(y(x)\), we get a circle of radius \(\cos x\). Now use polar coordinates (in the yz-plane) to get
Since \((u = x)\) and \((r = \cos x)\), we can substitute \((\cos u)\) for \((r)\) in the above equation to get
\[
\textbf{r}(u,v) = u \hat{i} + \cos u \cos v \hat{j} + \cos u \sin v \hat{k}.
\]

Normal Vectors and Tangent Planes

We have already learned how to find a normal vector of a surface that is presented as a function of two variables, namely find the gradient vector. To find the normal vector to a surface \(\textbf{r}(t)\) that is defined parametrically, we proceed as follows.

The partial derivatives
\[
\textbf{r}_u (u_0,v_0) \quad \text{and} \quad \textbf{r}_v (u_0,v_0)
\]
will lie on the tangent plane to the surface at the point \((u_0,v_0)\). This is true, because fixing one variable constant and letting the other vary, produced a curve on the surface through \((u_0,v_0)\). \(\textbf{r}_u (u_0,v_0)\) will be tangent to this curve. The tangent plane contains all vectors tangent to curves passing through the point.

To find a normal vector, we just cross the two tangent vectors.

Example \((\PageIndex{4})\)

Find the equation of the tangent plane to the surface
\[
\textbf{r} (u,v) = (u^2-v^2) \hat{i} + (u+v) \hat{j} + (uv) \hat{k}
\]
at the point \((1,2)\).

Solution

We have
\[
\textbf{r}_u (u,v) = (2u) \hat{i} + \hat{j} + v \hat{k}
\]
and
\[
\textbf{r}_v (u,v) = -2v \hat{i} + \hat{j} + u \hat{k}
\]
Therefore, the normal vector is
\[
\textbf{n}(1,2) = \textbf{r}_u (1,2) \times \textbf{r}_v (1,2) = (2 \hat{i} - \hat{j} + 2 \hat{k})
\]
\[ \textbf{r}_v (u,v) = (-2v) \hat{\textbf{i}} + \hat{\textbf{j}} + u \hat{\textbf{k}} \]

so that

\[ \textbf{r}_u (1,2) = 2 \hat{\textbf{i}} + \hat{\textbf{j}} + 2 \hat{\textbf{k}} \]
\[ \textbf{r}_v (1,2) = -4 \hat{\textbf{i}} + \hat{\textbf{j}} + \hat{\textbf{k}} \]
\[ \textbf{r}(1,2) = -3 \hat{\textbf{i}} + 3 \hat{\textbf{j}} + 3 \hat{\textbf{k}}. \]

Now cross these vectors together to get

\[
\begin{align}
\textbf{r}_u \times \textbf{r}_v &= \begin{vmatrix}
\hat{\textbf{i}} & \hat{\textbf{j}} & \hat{\textbf{k}} \\
2 & 1 & 2 \\
-4 & 1 & 1
\end{vmatrix} \\
&= - \hat{\textbf{i}} - 10 \hat{\textbf{j}} + 6 \hat{\textbf{k}}.
\end{align}
\]

We now have the normal vector and a point \((-3,3,2)). We use the normal vector-point equation for a plane

\[-1(x+3) - 10(y-3) + 6(z-2)=0 \]
\[-x-10y+6z=-15 \; \text{or} \; x+10y-6z=15.\]

**Surface Area**

To find the surface area of a parametrically defined surface, we proceed in a similar way as in the case as a surface defined by a function. Instead of projecting down to the region in the xy-plane, we project back to a region in the uv-plane. We cut the region into small rectangles which map approximately to small parallelograms with adjacent defining vectors \(\textbf{r}_u\) and \(\textbf{r}_v\). The area of these parallelograms will equal the magnitude of the cross product of \(\textbf{r}_u\) and \(\textbf{r}_v\). Finally add the areas up and take the limit as the rectangles get small. This will produce a double integral.

**Definition: Area of a Parametric Surface**

Let \(\mathcal{S}\) be a smooth surface defined parametrically by

\[ \textbf{r}(u,v) = x(u,v) \hat{\textbf{i}} + y(u,v) \hat{\textbf{j}} + z(u,v) \hat{\textbf{k}} \]

where \(u\) and \(v\) are contained in a region \((\text{mathbb}{\{R\}}))\). Then the surface area of \(\mathcal{S}\) is given by

\[ \text{SA} = \int_{\mathcal{R}} \|\textbf{r}_u \times \textbf{r}_v\| \; du \; dv.\]

Since the magnitude of a cross product involves a square root, the integral in the surface area formula is usually impossible or nearly impossible to evaluate without power series or by approximation techniques.

**Example**
Find the surface area of the surface given by

\[ \mathbf{r}(u,v) = (v^2) \hat{i} + (u-v) \hat{j} + (u^2) \hat{k}; \quad 0 \leq u \leq 2; \quad 1 \leq v \leq 4. \]

**Solution**

We calculate

\[ \mathbf{r}_u (u,v) = \hat{j} + 2u \hat{k} \]
\[ \mathbf{r}_v (u,v) = (2v) \hat{i} + \hat{j}. \]

The cross product is

\[
\begin{align}
||\mathbf{r} \times \mathbf{r} || &= \begin{vmatrix} 
\hat{i} & \hat{j} & \hat{k} \\
0 & 1 & 2u \\
2v & -1 & 0 
\end{vmatrix} \\
&= ||2u \hat{i} + 4uv \hat{j} - 2v \hat{k} || \\
&= 2\sqrt{u^2 + 4u^2v^2 + v^2}. 
\end{align}
\]

The surface area formula gives

\[ \text{SA} = \int_0^2 \int_1^4 2\sqrt{ 4u^2v^2 + v^2 } \; dv \; du. \]

This integral is probably impossible to compute exactly. Instead, a calculator can be used to obtain a surface area of 70.9.

Larry Green (Lake Tahoe Community College)