4.11: Hyperbolic Functions

The hyperbolic functions appear with some frequency in applications, and are quite similar in many respects to the trigonometric functions. This is a bit surprising given our initial definitions.

Definition 4.11.1: Hyperbolic Cosines and Sines

The hyperbolic cosine is the function

\[
\cosh x = \frac{e^x + e^{-x}}{2},
\]

and the hyperbolic sine is the function

\[
\sinh x = \frac{e^x - e^{-x}}{2}.
\]

Notice that \(\cosh\) is even (that is, \(\cosh(-x) = \cosh(x)\)) while \(\sinh\) is odd (\(\sinh(-x) = -\sinh(x)\)), and \(\cosh x + \sinh x = e^x\). Also, for all \(x\), \(\cosh x > 0\), while \(\sinh x = 0\) if and only if \(e^x - e^{-x} = 0\), which is true precisely when \(x = 0\).

Lemma 4.11.2

The range of \(\cosh x\) is \([1, \infty)\).

Proof

Let \(y = \cosh x\). We solve for \(x\):
\[
\begin{align*}
y &= \frac{e^x + e^{-x}}{2} \\
2y &= e^x + e^{-x} \\
2ye^x &= e^{2x} + 1 \\
0 &= e^{2x} - 2ye^x + 1 \\
e^x &= \frac{2y \pm \sqrt{4y^2 - 4}}{2} \\
e^x &= y \pm \sqrt{y^2 - 1}
\end{align*}
\]

From the last equation, we see \( y^2 \geq 1 \), and since \( y \geq 0 \), it follows that \( y \geq 1 \).

Now suppose \( y \geq 1 \), so \( y \pm \sqrt{y^2 - 1} > 0 \). Then \( x = \ln(y \pm \sqrt{y^2 - 1}) \) is a real number, and \( y = \cosh x \), so \( y \) is in the range of \( \cosh(x) \).

\( \square \)

**Definition 4.11.3: Hyperbolic Tangent and Cotangent**

The other hyperbolic functions are

\[
\begin{align*}
\tanh x &= \frac{\sinh x}{\cosh x} \\
\coth x &= \frac{\cosh x}{\sinh x} \\
\text{sech} x &= \frac{1}{\cosh x} \\
\text{csch} x &= \frac{1}{\sinh x}
\end{align*}
\]

The domain of \( \coth \) and \( \text{csch} \) is \( x \neq 0 \) while the domain of the other hyperbolic functions is all real numbers. Graphs are shown in Figure \( \PageIndex{1} \)

\( \square \)

Certainly the hyperbolic functions do not closely resemble the trigonometric functions graphically. But they do have analogous properties, beginning with the following identity.

**Theorem 4.11.4**

For all \( x \in \mathbb{R} \), \( \cosh^2 x - \sinh^2 x = 1 \).

**Proof**

The proof is a straightforward computation:

\[
\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = \frac{4}{4} = 1.
\]

\( \square \)

This immediately gives two additional identities:
\[1 - \tanh^2 x = \text{sech}^2 x \quad \hbox{and} \quad \coth^2 x - 1 = \text{csch}^2 x.\]

The identity of the theorem also helps to provide a geometric motivation. Recall that the graph of \(x^2 - y^2 = 1\) is a hyperbola with asymptotes \(x = \pm y\) whose \(x\)-intercepts are \(\pm 1\). If \((x, y)\) is a point on the right half of the hyperbola, and if we let \(x = \cosh t\), then \(y = \sqrt{x^2 - 1} = \pm \sqrt{\cosh^2 t - 1} = \pm \sinh t\). So for some suitable \(t\), \(\cosh t\) and \(\sinh t\) are the coordinates of a typical point on the hyperbola. In fact, it turns out that \(t\) is twice the area shown in the first graph of Figure \(\PageIndex{2}\). Even this is analogous to trigonometry; \(\cos t\) and \(\sin t\) are the coordinates of a typical point on the unit circle, and \(t\) is twice the area shown in the second graph of Figure \(\PageIndex{2}\).

**Figure \(\PageIndex{2}\):** Geometric definitions of \(\sin, \cos, \sinh, \cosh\): \(t\) is twice the shaded area in each figure.

Given the definitions of the hyperbolic functions, finding their derivatives is straightforward. Here again we see similarities to the trigonometric functions.

**Theorem 4.11.5**

\[\frac{d}{dx} \cosh x = \sinh x\]

and

\[\frac{d}{dx} \sinh x = \cosh x.\]

**Proof**

\[\frac{d}{dx} \cosh x = \frac{d}{dx} \left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x,\]

and

\[\frac{d}{dx} \sinh x = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x.\]

\(\square\)

Since \(\cosh x > 0\), \(\sinh x\) is increasing and hence injective, so \(\sinh x\) has an inverse, \(\arcsinh x\). Also, \(\sinh x > 0\) when \(x > 0\), so \(\cosh x\) is injective on \((0, \infty)\) and has a (partial) inverse, \(\text{arccosh} x\). The other hyperbolic functions have inverses as well, though \(\text{arccsch} x\) is only a partial inverse. We may compute the derivatives of these functions as we have other inverse functions.

**Theorem 4.11.6**

\[\frac{d}{dx} \text{arcsinh} x = \frac{1}{\sqrt{1 + x^2}}\]
Proof

Let \( y = \text{arcsinh} \, x \), so \( \sinh y = x \). Then

\[
\frac{d}{dx} \sinh y = \cosh(y) \cdot y' = 1,
\]

and so

\[
y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}.
\]

\( \square \)

The other derivatives are left to the exercises.

Contributors

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