4.11: Hyperbolic Functions

The hyperbolic functions appear with some frequency in applications, and are quite similar in many respects to the trigonometric functions. This is a bit surprising given our initial definitions.

Definition 4.11.1: Hyperbolic Cosines and Sines

The hyperbolic cosine is the function

\[ \cosh x = \frac{e^x + e^{-x}}{2}, \]

and the hyperbolic sine is the function

\[ \sinh x = \frac{e^x - e^{-x}}{2}. \]

Notice that \( \cosh \) is even (that is, \( \cosh(-x) = \cosh(x) \)) while \( \sinh \) is odd (\( \sinh(-x) = -\sinh(x) \)), and \( \cosh x + \sinh x = e^x \). Also, for all \( x \), \( \cosh x > 0 \), while \( \sinh x = 0 \) if and only if \( e^x - e^{-x} = 0 \), which is true precisely when \( x = 0 \).

Lemma 4.11.2

The range of \( \cosh x \) is \( [1, \infty) \).

Proof

Let \( y = \cosh x \). We solve for \( x \):
\[
\begin{align*}
y &= \frac{e^x + e^{-x}}{2} \\
2y &= e^x + e^{-x} \\
2ye^x &= e^{2x} + 1 \\
0 &= e^{2x} - 2ye^x + 1 \\
e^x &= \frac{2y \pm \sqrt{4y^2 - 4}}{2} \\
e^x &= y \pm \sqrt{y^2 - 1}
\end{align*}
\]

From the last equation, we see \( y^2 \geq 1 \), and since \( y \geq 0 \), it follows that \( y \geq 1 \).

Now suppose \( y \geq 1 \), so \( y \pm \sqrt{y^2 - 1} > 0 \). Then \( x = \ln(y \pm \sqrt{y^2 - 1}) \) is a real number, and \( y = \cosh x \), so \( y \) is in the range of \( \cosh(x) \).

\( \square \)

Definition 4.11.3: Hyperbolic Tangent and Cotangent

The other hyperbolic functions are

\[
\begin{align*}
\tanh x &= \frac{\sinh x}{\cosh x} \\
\coth x &= \frac{\cosh x}{\sinh x} \\
\text{sech} x &= \frac{1}{\cosh x} \\
\text{csch} x &= \frac{1}{\sinh x}
\end{align*}
\]

The domain of \( \coth \) and \( \text{csch} \) is \( x \neq 0 \) while the domain of the other hyperbolic functions is all real numbers. Graphs are shown in Figure \( \PageIndex{1} \).

**Figure \( \PageIndex{1} \): The hyperbolic functions.**

Certainly the hyperbolic functions do not closely resemble the trigonometric functions graphically. But they do have analogous properties, beginning with the following identity.

**Theorem 4.11.4**

For all \( x \in \mathbb{R} \), \( \cosh^2 x - \sinh^2 x = 1 \).

**Proof**

The proof is a straightforward computation:

\[
\text{cosh}^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = \frac{4}{4} = 1.
\]

\( \square \)

This immediately gives two additional identities:
$[1-\tanh^2 x = \text{sech}^2 x \quad \text{and} \quad \coth^2 x - 1 = \text{csch}^2 x]$

The identity of the theorem also helps to provide a geometric motivation. Recall that the graph of $x^2 - y^2 = 1$ is a hyperbola with asymptotes $x = \pm y$ whose $x$-intercepts are $\pm 1$. If $(x, y)$ is a point on the right half of the hyperbola, and if we let $x = \cosh t$, then $y = \pm \sqrt{x^2 - 1} = \pm \sqrt{\cosh^2 x - 1} = \pm \sinh t$. So for some suitable $t$, $\cosh t$ and $\sinh t$ are the coordinates of a typical point on the hyperbola. In fact, it turns out that $t$ is twice the area shown in the first graph of Figure 2. Even this is analogous to trigonometry; $\cos t$ and $\sin t$ are the coordinates of a typical point on the unit circle, and $t$ is twice the area shown in the second graph of Figure 2.

**Figure 2:** Geometric definitions of sin, cos, sinh, cosh: $t$ is twice the shaded area in each figure.

Given the definitions of the hyperbolic functions, finding their derivatives is straightforward. Here again we see similarities to the trigonometric functions.

**Theorem 4.11.5**

$\frac{d}{dx} \cosh x = \sinh x$ and $\frac{d}{dx} \sinh x = \cosh x$.

**Proof**

$\frac{d}{dx} \cosh x = \frac{d}{dx} \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x,$

and

$\frac{d}{dx} \sinh x = \frac{d}{dx} \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x.$

$\square$

Since $\cosh x > 0$, $\sinh x$ is increasing and hence injective, so $\sinh x$ has an inverse, $\arcsinh x$. Also, $\sinh x > 0$ when $x > 0$, so $\cosh x$ is injective on $(0, \infty)$ and has a (partial) inverse, $\text{arccosh} x$. The other hyperbolic functions have inverses as well, though $\ln \text{arccosech} x$ is only a partial inverse. We may compute the derivatives of these functions as we have other inverse functions.

**Theorem 4.11.6**

$\frac{d}{dx} \text{arcsinh} x = \frac{1}{\sqrt{x^2 + 1}}$.

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Proof

Let \( y = \text{arcsinh} \ x \), so \( \sinh y = x \). Then

\[
\frac{d}{dx} \sinh y = \cosh(y) \cdot y' = 1,
\]

and so

\[
\frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}.
\]

\( \square \)

The other derivatives are left to the exercises.

Contributors

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