7.1: Rational and Irrational Numbers

Skills to Develop

- Identify rational numbers and irrational numbers
- Classify different types of real numbers

be prepared!

Before you get started, take this readiness quiz.

1. Write 3.19 as an improper fraction. If you missed this problem, review Example 5.1.4.
2. Write $\frac{5}{11}$ as a decimal. If you missed this problem, review Example 5.5.3.
3. Simplify: $\sqrt{144})$. If you missed this problem, review Example 5.12.1.

Identify Rational Numbers and Irrational Numbers

Congratulations! You have completed the first six chapters of this book! It's time to take stock of what you have done so far in this course and think about what is ahead. You have learned how to add, subtract, multiply, and divide whole numbers, fractions, integers, and decimals. You have become familiar with the language and symbols of algebra, and have simplified and evaluated algebraic expressions. You have solved many different types of applications. You have established a good solid foundation that you need so you can be successful in algebra.

In this chapter, we'll make sure your skills are firmly set. We'll take another look at the kinds of numbers we have worked with in all previous chapters. We'll work with properties of numbers that will help you improve your number sense. And we'll
practice using them in ways that we’ll use when we solve equations and complete other procedures in algebra.

We have already described numbers as counting numbers, whole numbers, and integers. Do you remember what the difference is among these types of numbers?

- **Counting numbers**: 1, 2, 3, 4…
- **Whole numbers**: 0, 1, 2, 3, 4…
- **Integers**: …−3, −2, −1, 0, 1, 2, 3, 4…

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**Rational Numbers**

What type of numbers would you get if you started with all the integers and then included all the fractions? The numbers you would have form the set of rational numbers. A *rational number* is a number that can be written as a ratio of two integers.

**Definition: Rational Numbers**

A rational number is a number that can be written in the form \(\frac{p}{q}\), where \(p\) and \(q\) are integers and \(q \neq 0\).

All fractions, both positive and negative, are rational numbers. A few examples are:

\[
\frac{4}{5}, -\frac{7}{8}, \frac{13}{4}, \text{ and } -\frac{20}{3}
\]

Each numerator and each denominator is an integer.

We need to look at all the numbers we have used so far and verify that they are rational. The definition of rational numbers tells us that all fractions are rational. We will now look at the counting numbers, whole numbers, integers, and decimals to make sure they are rational.

Are integers rational numbers? To decide if an integer is a rational number, we try to write it as a ratio of two integers. An easy way to do this is to write it as a fraction with denominator one.

\[
3 = \frac{3}{1}, \quad -8 = \frac{-8}{1}, \quad 0 = \frac{0}{1}
\]

Since any integer can be written as the ratio of two integers, all integers are rational numbers. Remember that all the counting numbers and all the whole numbers are also integers, and so they, too, are rational.

What about decimals? Are they rational? Let’s look at a few to see if we can write each of them as the ratio of two integers. We’ve already seen that integers are rational numbers. The integer −8 could be written as the decimal −8.0. So, clearly, some decimals are rational.

Think about the decimal 7.3. Can we write it as a ratio of two integers? Because 7.3 means \(7 \frac{3}{10}\), we can write it as an improper fraction, \(\frac{73}{10}\). So 7.3 is the ratio of the integers 73 and 10. It is a rational number.
In general, any decimal that ends after a number of digits (such as 7.3 or −1.2684) is a rational number. We can use the reciprocal (or multiplicative inverse) of the place value of the last digit as the denominator when writing the decimal as a fraction.

Example \(\PageIndex{1}\):

Write each as the ratio of two integers: (a) −15 (b) 6.81 (c) \((-3 \frac{6}{7})\).

**Solution**

(a) −15

Write the integer as a fraction with denominator 1. \(\dfrac{-15}{1}\)

(b) 6.81

Write the decimal as a mixed number. \(6 \dfrac{81}{100}\)

Then convert it to an improper fraction. \(\dfrac{681}{100}\)

(c) \((-3 \frac{6}{7})\)

Convert the mixed number to an improper fraction. \(-\dfrac{27}{7}\)

Exercise \(\PageIndex{1}\):

Write each as the ratio of two integers: (a) −24 (b) 3.57.

**Answer a**

\(\dfrac{-24}{1}\)

**Answer b**

\(\dfrac{357}{100}\)

Exercise \(\PageIndex{2}\):

Write each as the ratio of two integers: (a) −19 (b) 8.41.

**Answer a**

\(\dfrac{-19}{1}\)
Let's look at the decimal form of the numbers we know are rational. We have seen that every integer is a rational number, since \( a = \frac{a}{1} \) for any integer, \( a \). We can also change any integer to a decimal by adding a decimal point and a zero.

\[
\begin{align*}
\text{Integer} & \quad -2, \quad -1, \quad 0, \quad 1, \quad 2, \quad 3 \\
\text{Decimal} & \quad -2.0, \quad -1.0, \quad 0.0, \quad 1.0, \quad 2.0, \quad 3.0
\end{align*}
\]

These decimal numbers stop.

We have also seen that every fraction is a rational number. Look at the decimal form of the fractions we just considered.

\[
\begin{align*}
\text{Ratio of Integers} & \quad \frac{4}{5}, \quad -\frac{7}{8}, \quad \frac{13}{4}, \quad -\frac{20}{3} \\
\text{Decimal forms} & \quad 0.8, \quad -0.875, \quad 3.25, \quad -6.66\overline{6}
\end{align*}
\]

These decimals either stop or repeat.

What do these examples tell you? Every rational number can be written both as a ratio of integers and as a decimal that either stops or repeats. The table below shows the numbers we looked at expressed as a ratio of integers and as a decimal.

**Rational Numbers**

<table>
<thead>
<tr>
<th>Fractions</th>
<th>Integers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>(\frac{4}{5}, \frac{-7}{8}, \frac{13}{4}, \frac{-20}{3})</td>
</tr>
<tr>
<td>Ratio of Integer</td>
<td>(\frac{4}{5}, \frac{-7}{8}, \frac{13}{4}, \frac{-20}{3})</td>
</tr>
<tr>
<td>Decimal number</td>
<td>(0.8, -0.875, 3.25, -6.\overline{6})</td>
</tr>
</tbody>
</table>

**Irrational Numbers**

Are there any decimals that do not stop or repeat? Yes. The number \(\pi\) (the Greek letter pi, pronounced ‘pie’), which is very important in describing circles, has a decimal form that does not stop or repeat.

\[
\pi = 3.141592654 \ldots
\]

Similarly, the decimal representations of square roots of whole numbers that are not perfect squares never stop and never
repeat. For example,
\[
\sqrt{5} = 2.236067978 \ldots
\]

A decimal that does not stop and does not repeat cannot be written as the ratio of integers. We call this kind of number an **irrational number**.

**Definition: Irrational Number**

An irrational number is a number that cannot be written as the ratio of two integers. Its decimal form does not stop and does not repeat.

Let's summarize a method we can use to determine whether a number is rational or irrational.

If the decimal form of a number

- stops or repeats, the number is rational.
- does not stop and does not repeat, the number is irrational.

**Example \(\PageIndex{2}\):**

Identify each of the following as rational or irrational: (a) 0.58\(\overline{3}\) (b) 0.475 (c) 3.60551275\(\ldots\)

**Solution**

(a) 0.58\(\overline{3}\)

The bar above the 3 indicates that it repeats. Therefore, 0.583\(\ldots\) is a repeating decimal, and is therefore a rational number.

(b) 0.475

This decimal stops after the 5, so it is a rational number.

(c) 3.60551275\(\ldots\)

The ellipsis (\(\ldots\)) means that this number does not stop. There is no repeating pattern of digits. Since the number doesn't stop and doesn't repeat, it is irrational.

**Exercise \(\PageIndex{3}\):**

Identify each of the following as rational or irrational: (a) 0.29 (b) 0.81\(\overline{6}\) (c) 2.515115111\(\ldots\)

**Answer a**

rational
Answer a
rational

Answer b

rational

Answer c
irrational

Let's think about square roots now. Square roots of perfect squares are always whole numbers, so they are rational. But the decimal forms of square roots of numbers that are not perfect squares never stop and never repeat, so these square roots are irrational.

Example:

Identify each of the following as rational or irrational: (a) 36 (b) 44

Solution

(a) The number 36 is a perfect square, since \( 6^2 = 36 \). So \( \sqrt{36} = 6 \). Therefore \( \sqrt{36} \) is rational.

(b) Remember that \( 6^2 = 36 \) and \( 7^2 = 49 \), so 44 is not a perfect square. This means \( \sqrt{44} \) is irrational.

Exercise:

Identify each of the following as rational or irrational: (a) \( \sqrt{81} \) (b) \( \sqrt{17} \)

Answer a
rational
Exercise \(\PageIndex{6}\):

Identify each of the following as rational or irrational: (a) \(\sqrt{116}\) (b) \(\sqrt{121}\)

**Answer a**

irrational

**Answer b**

rational

---

**Classify Real Numbers**

We have seen that all counting numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. Irrational numbers are a separate category of their own. When we put together the rational numbers and the irrational numbers, we get the set of real numbers. Figure \(\PageIndex{1}\) illustrates how the number sets are related.

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**Definition: Real Numbers**

Real numbers are numbers that are either rational or irrational.

Does the term “real numbers” seem strange to you? Are there any numbers that are not “real”, and, if so, what could they be? For centuries, the only numbers people knew about were what we now call the real numbers. Then mathematicians discovered the set of imaginary numbers. You won't encounter imaginary numbers in this course, but you will later on in your studies of algebra.
Example \(\PageIndex{4}\):

Determine whether each of the numbers in the following list is a (a) whole number, (b) integer, (c) rational number, (d) irrational number, and (e) real number.

\([-7, \dfrac{14}{5}, 8, \sqrt{5}, 5.9, -\sqrt{64}\]

**Solution**

a. The whole numbers are 0, 1, 2, 3,… The number 8 is the only whole number given.

b. The integers are the whole numbers, their opposites, and 0. From the given numbers, \(-7\) and 8 are integers. Also, notice that 64 is the square of 8 so \((-\sqrt{64}) = -8\). So the integers are \(-7, 8, -\sqrt{64}\).

c. Since all integers are rational, the numbers \(-7, 8,\) and \((-\sqrt{64})\) are also rational. Rational numbers also include fractions and decimals that terminate or repeat, so \(\dfrac{14}{5}\) and 5.9 are rational.

d. The number 5 is not a perfect square, so \(\sqrt{5}\) is irrational.

e. All of the numbers listed are real.

We’ll summarize the results in a table.

<table>
<thead>
<tr>
<th>Number</th>
<th>Whole</th>
<th>Integer</th>
<th>Rational</th>
<th>Irrational</th>
<th>Real</th>
</tr>
</thead>
<tbody>
<tr>
<td>-7</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>(\dfrac{14}{5})</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>8</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>(\sqrt{5})</td>
<td></td>
<td></td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>5.9</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>(-\sqrt{64})</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

Exercise \(\PageIndex{7}\):

Determine whether each number is a (a) whole number, (b) integer, (c) rational number, (d) irrational number, and (e) real number: \(-3, -\sqrt{2}, 0.\overline{3}, \dfrac{9}{5}, 4, \sqrt{49}\).
### Answer

<table>
<thead>
<tr>
<th>Number</th>
<th>Whole</th>
<th>Integer</th>
<th>Rational</th>
<th>Irrational</th>
<th>Real</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td></td>
<td></td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>(-\sqrt{2})</td>
<td></td>
<td></td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>(0.\overline{3})</td>
<td></td>
<td></td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>(\frac{9}{5})</td>
<td></td>
<td></td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>(4)</td>
<td></td>
<td></td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>(\sqrt{49})</td>
<td></td>
<td></td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>(2.041975…)</td>
<td></td>
<td></td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
</tbody>
</table>

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### Exercise \(\PageIndex{8}\):

Determine whether each number is a (a) whole number, (b) integer, (c) rational number, (d) irrational number, and (e) real number: \((-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}, 2.041975…\)

### Answer

<table>
<thead>
<tr>
<th>Number</th>
<th>Whole</th>
<th>Integer</th>
<th>Rational</th>
<th>Irrational</th>
<th>Real</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\sqrt{25})</td>
<td></td>
<td></td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>(-\frac{3}{8})</td>
<td></td>
<td></td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>(-1)</td>
<td></td>
<td></td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>(6)</td>
<td></td>
<td></td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>(\sqrt{121})</td>
<td></td>
<td></td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>(2.041975…)</td>
<td></td>
<td></td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
</tbody>
</table>

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### ACCESS ADDITIONAL ONLINE RESOURCES

- **Sets of Real Numbers**
- **Real Numbers**

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**Practice Makes Perfect**
Rational Numbers

In the following exercises, write as the ratio of two integers.

1. (a) 5 (b) 3.19
2. (a) 8 (b) −1.61
3. (a) −12 (b) 9.279
4. (a) −16 (b) 4.399

In the following exercises, determine which of the given numbers are rational and which are irrational.

5. 0.75, 0.22\(\overline{3}\), 1.39174…
6. 0.36, 0.94729…, 2.52\(\overline{8}\)
7. 0.\(\overline{45}\), 1.919293…, 3.59
8. 0.1\(\overline{3}\), 0.42982…, 1.875

In the following exercises, identify whether each number is rational or irrational.

9. (a) 25 (b) 30
10. (a) 44 (b) 49
11. (a) 164 (b) 169
12. (a) 225 (b) 216

Classifying Real Numbers

In the following exercises, determine whether each number is whole, integer, rational, irrational, and real.

13. −8, 0, 1.95286…, \(\sqrt[12]{5}\), \(\sqrt{36}\), 9
14. −9, \(\sqrt[3]{9}\), −\(\sqrt{9}\), 0.4\(\overline{09}\), \(\sqrt[11]{6}\), 7
15. \(\sqrt[100]{100}\), −7, \(\sqrt[8]{3}\), −1, 0.77, \(3\sqrt[4]{1}\)

Everyday Math

16. **Field trip** All the 5th graders at Lincoln Elementary School will go on a field trip to the science museum. Counting all the children, teachers, and chaperones, there will be 147 people. Each bus holds 44 people.
   a. How many buses will be needed?
   b. Why must the answer be a whole number?
   c. Why shouldn't you round the answer the usual way?

17. **Child care** Serena wants to open a licensed child care center. Her state requires that there be no more than 12 children for each teacher. She would like her child care center to serve 40 children.
   a. How many teachers will be needed?
b. Why must the answer be a whole number?

c. Why shouldn't you round the answer the usual way?

Writing Exercises

18. In your own words, explain the difference between a rational number and an irrational number.

19. Explain how the sets of numbers (counting, whole, integer, rational, irrationals, reals) are related to each other.

Self Check

(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

<table>
<thead>
<tr>
<th>I can...</th>
<th>Confidently</th>
<th>With some help</th>
<th>No-I don't get it!</th>
</tr>
</thead>
<tbody>
<tr>
<td>identify rational and irrational numbers.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>classify different types of real numbers.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no—I don’t get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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