7.2: Exponential Change and Separable Differential Equations

Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

• What is a separable differential equation?
• How can we find solutions to a separable differential equation?
• Are some of the differential equations that arise in applications separable?

In Sections 7.2 and 7.3, we have seen several ways to approximate the solution to an initial value problem. Given the frequency with which differential equations arise in the world around us, we would like to have some techniques for finding explicit algebraic solutions of certain initial value problems. In this section, we focus on a particular class of differential equations (called separable) and develop a method for finding algebraic formulas for solutions to these equations.

A separable differential equation is a differential equation whose algebraic structure permits the variables present to be separated in a particular way. For instance, consider the equation

\[
\frac{dy}{dt} = ty.
\]

We would like to separate the variables \(t\) and \(y\) so that all occurrences of \(t\) appear on the right-hand side, and all occurrences of \(y\) appears on the left and multiply \(dy/dt\). We may do this in the preceding differential equation by dividing both sides by \(y\):

\[
\frac{1}{y} \frac{dy}{dt} = t.
\]

Note particularly that when we attempt to separate the variables in a differential equation, we require that the left-hand side be
a product in which the derivative \( \frac{dy}{dt} \) is one term. Not every differential equation is separable. For example, if we consider the equation

\[
\frac{dy}{dt} = t - y,
\]

it may seem natural to separate it by writing

\[
y + \frac{dy}{dt} = t.
\]

As we will see, this will not be helpful since the left-hand side is not a product of a function of \( y \) with \( \frac{dy}{dt} \).

Preview Activity \( \PageIndex{1} \):

In this preview activity, we explore whether certain differential equations are separable or not, and then revisit some key ideas from earlier work in integral calculus.

a. Which of the following differential equations are separable? If the equation is separable, write the equation in the revised form \( g(y) \frac{dy}{dt} = h(t) \).

1. \( \frac{dy}{dt} = -3y \)
2. \( \frac{dy}{dt} t = ty - y. \)
3. \( \frac{dy}{dt} t = t + 1. \)
4. \( \frac{dy}{dt} t = t^2 - y^2. \)

b. Explain why any autonomous differential equation is guaranteed to be separable.

c. Why do we include the term "\(+C\)" in the expression \( \int x \, dx = \frac{x^2}{2} + C \)?

d. Suppose we know that a certain function \( f \) satisfies the equation \( \int f'(x) \, dx = \int x \, dx \).

What can you conclude about \( f \)?

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Solving separable differential equations

Before we discuss a general approach to solving a separable differential equation, it is instructive to consider an example.

Example \( \PageIndex{1} \): Separable Differential Equation

Find all functions \( y \) that are solutions to the differential equation

\[
\frac{dy}{dt} = \frac{t}{y^2}.
\]

Solution

We begin by separating the variables and writing

\[
y^2 \frac{dy}{dt} = t.
\]
Integrating both sides of the equation with respect to the independent variable \( t \) shows that

\[
\int y^2 \, \text{d}y \, \text{d}t = \int t \, \text{d}t. \quad \text{(nonumber)}
\]

Next, we notice that the left-hand side allows us to change the variable of antidifferentiation from \( t \) to \( y \). In particular,

\[
\text{d}y = \frac{\text{d}y}{\text{d}t} \, \text{d}t,
\]
so we now have

\[
\int y^2 \, \text{d}y = \int t \, \text{d}t. \quad \text{(nonumber)}
\]

This most recent equation says that two families of antiderivatives are equal to one another. Therefore, when we find representative antiderivatives of both sides, we know they must differ by arbitrary constant \( C \). Antidifferentiating and including the integration constant \( C \) on the right, we find that

\[
\int \frac{y^3}{3} \, \text{d}y = \int \frac{t^2}{2} \, \text{d}t + C. \quad \text{(nonumber)}
\]

Again, note that it is not necessary to include an arbitrary constant on both sides of the equation; we know that \( \int \frac{y^3}{3} \, \text{d}y \) and \( \int \frac{t^2}{2} \, \text{d}t \) are in the same family of antiderivatives and must therefore differ by a single constant.

Finally, we may now solve the last equation above for \( y \) as a function of \( t \), which gives

\[
y(t) = \sqrt[3]{\frac{3}{2} t^2 + 3C}. \quad \text{(nonumber)}
\]

Of course, the term \( 3C \) on the right-hand side represents 3 times an unknown constant. It is, therefore, still an unknown constant, which we will rewrite as \( C \). We thus conclude that the function

\[
y(t) = \sqrt[3]{\frac{3}{2} t^2 + C}. \quad \text{(nonumber)}
\]

is a solution to the original differential equation for any value of \( C \).

Notice that because this solution depends on the arbitrary constant \( C \), we have found an infinite family of solutions. This makes sense because we expect to find a unique solution that corresponds to any given initial value.

For example, if we want to solve the initial value problem

\[
\frac{\text{d}y}{\text{d}t} = \frac{t}{y^2} \quad \text{nonumber}
\]

with \( y(0) = 2 \),

we know that the solution has the form

\[
y(t) = \sqrt[3]{\frac{3}{2} t^2 + 3C}. \quad \text{(nonumber)}
\]

for some constant \( C \). We therefore must find the appropriate value for \( C \) that gives the initial value \( y(0) = 2 \). Hence,

\[
y(0) = 2 = \sqrt[3]{\frac{3}{2} t^2 + 3C} \quad \text{for some constant \( C \).}
\]

which shows that \( C = 2^3 = 8 \). The solution to the initial value problem is then
The strategy of Example \(\PageIndex{1}\) may be applied to any differential equation of the form 
\[
\frac{dy}{dt} = g(y) \cdot h(t),
\]
and any differential equation of this form is said to be separable. We work to solve a separable differential equation by writing
\[
\frac{1}{g(y)} \frac{dy}{dt} = h(t),
\]
and then integrating both sides with respect to \(t\). After integrating, we strive to solve algebraically for \(y\) in order to write \(y\) as a function of \(t\).

We consider one more example before doing further exploration in some activities.

Example \(\PageIndex{2}\): Separable Differential Equation

Solve the differential equation
\[
\frac{dy}{dt} = 3y.
\]

\textbf{Solution}

Following the same strategy as in Example \(\PageIndex{1}\), we have
\[
\frac{1}{y} \frac{dy}{dt} = 3.
\]

Integrating both sides with respect to \(t\),
\[
\int \frac{1}{y} \, dy = \int 3 \, dt,
\]
and thus
\[
\ln |y| = 3t + C.
\]

Antidifferentiating and including the integration constant, we find that
\[
\ln |y| = 3t + C.
\]

Finally, we need to solve for \(y\). Here, one point deserves careful attention. By the definition of the natural logarithm function, it follows that
\[
|y| = e^{3t+C} = e^{3t} e^C.
\]

Since \(C\) is an unknown constant, \(e^C\) is as well, though we do know that it is positive (because \(e^x\) is positive for any \(x\)). When we remove the absolute value in order to solve for \(y\), however, this constant may be either positive or negative. We will denote this updated constant (that accounts for a possible + or −) by \(C\) to obtain
\[
y(t) = Ce^{3t}.
\]
There is one more slightly technical point to make. Notice that \( y = 0 \) is an **equilibrium solution** to this differential equation. In solving the equation above, we begin by dividing both sides by \( y \), which is not allowed if \( y = 0 \). To be perfectly careful, therefore, we will typically consider the equilibrium solutions separably. In this case, notice that the final form of our solution captures the equilibrium solution by allowing \( C = 0 \).

Activity \( \PageIndex{1} \)

Suppose that the population of a town is growing continuously at an annual rate of 3% per year.

a. Let \( P(t) \) be the population of the town in year \( t \). Write a differential equation that describes the annual growth rate.

b. Find the solutions of this differential equation.

c. If you know that the town’s population in year 0 is 10,000, find the population \( P(t) \).

d. How long does it take for the population to double? This time is called the doubling time.

e. Working more generally, find the doubling time if the annual growth rate is \( k \) times the population.

Activity \( \PageIndex{2} \): Cooling Coffee

Suppose that a cup of coffee is initially at a temperature of 105° F and is placed in a 75° F room. Newton’s law of cooling says that

\[
\frac{dT}{dt} = -k(T - 75),
\]

where \( k \) is a constant of proportionality.

a. Suppose you measure that the coffee is cooling at one degree per minute at the time the coffee is brought into the room. Use the differential equation to determine the value of the constant \( k \).

b. Find all the solutions of this differential equation.

c. What happens to all the solutions as \( t \to \infty \)? Explain how this agrees with your intuition.

d. What is the temperature of the cup of coffee after 20 minutes?

e. How long does it take for the coffee to cool to 80°?

Activity \( \PageIndex{3} \)

Solve each of the following differential equations or initial value problems.

a. \[
\frac{dy}{dt} - (2 - t)y = 2 - t
\]

b. \[
\frac{1}{t} \frac{dy}{dt} = e^{(t^2 - 2y)}
\]

c. \[
y' = 2y + 2, \ y(0) = 2
\]

d. \[
y' = 2y^2, \ y(-1) = 2
\]

e. \[
\frac{dy}{dt} = \frac{-2ty}{t^2 + 1}, \ y(0) = 4
\]
Summary

In this section, we encountered the following important ideas:

- A separable differential equation is one that may be rewritten with all occurrences of the dependent variable multiplying the derivative and all occurrences of the independent variable on the other side of the equation.
- We may find the solutions to certain separable differential equations by separating variables, integrating with respect to $t(t)$, and ultimately solving the resulting algebraic equation for $y(y)$.
- This technique allows us to solve many important differential equations that arise in the world around us. For instance, questions of growth and decay and Newton’s Law of Cooling give rise to separable differential equations. Later, we will learn in Section 7.6 that the important logistic differential equation is also separable.

Contributors and Attributions

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