4.1: Differentiation and Integration of Vector Valued Functions

The formal definition of the derivative of a vector valued function is very similar to the definition of the derivative of a real valued function.

Definition: The Derivative of a Vector Valued Function

Let \( r(t) \) be a vector valued function, then

\[
\frac{d}{dt} \left( \sum_{i} a_i \mathbf{e}_i \right) = \sum_{i} \frac{da_i}{dt} \mathbf{e}_i,
\]

Where \( a_i \) are the components of the vector valued function. Because the derivative of a sum is the sum of the derivative, we can find the derivative of each of the components of the vector valued function to find its derivative.

Example

\[
\frac{d}{dt} (3 \mathbf{i} + \sin t \mathbf{j}) = \cos t \mathbf{j}
\]

\[
\frac{d}{dt} \left(3t^2 \mathbf{i} + \cos{(4t)} \mathbf{j} + te^t \mathbf{k} \right) = 6t \mathbf{i} -4 \sin{(t)} \mathbf{j} + (e^t + te^t) \mathbf{k}
\]

Properties of Vector Valued Functions

All of the properties of differentiation still hold for vector values functions. Moreover because there are a variety of ways of defining multiplication, there is an abundance of product rules.

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Suppose that \( \mathbf{v}(t) \) and \( \mathbf{w}(t) \) are vector valued functions, \( f(t) \) is a scalar function, and \( c \) is a real number then

1. \( \frac{d}{dt} (\mathbf{v}(t) + \mathbf{w}(t)) = \frac{d}{dt} \mathbf{v}(t) + \frac{d}{dt} \mathbf{w}(t) \),
2. \( \frac{d}{dt} c \mathbf{v}(t) = c \frac{d}{dt} \mathbf{v}(t) \),
3. \( \frac{d}{dt} (f(t) \mathbf{v}(t)) = f'(t) \mathbf{v}(t) + f(t) \mathbf{v}(t)' \),
4. \( (\mathbf{v}(t) \cdot \mathbf{w}(t))' = \mathbf{v}'(t) \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \mathbf{w}'(t) \),
5. \( (\mathbf{v}(t) \times \mathbf{w}(t))' = \mathbf{v}'(t) \times \mathbf{w}(t) + \mathbf{v}(t) \times \mathbf{w}'(t) \),
6. \( \frac{d}{dt} \mathbf{v}(f(t)) = \mathbf{v}(t)'(f(t)) f'(t) \).

**Example \( \PageIndex{2} \)**

Show that if \( \mathbf{r} \) is a differentiable vector valued function with constant magnitude, then

\[
\mathbf{r} \cdot \mathbf{r}' = 0.
\]

**Solution**

Since \( \mathbf{r} \) has constant magnitude, call its magnitude \( \|k\| \),

\[
\|k^2 = \|r^2 = r \cdot r \|.
\]

Taking derivatives of the left and right sides gives

\[
0 = (\mathbf{r} \cdot \mathbf{r})' = r' \cdot \mathbf{r} + r \cdot \mathbf{r}'
\]

\[
= r \cdot \mathbf{r}' + r \cdot \mathbf{r}' = 2r \cdot \mathbf{r}'.
\]

Divide by two and the result follows

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**Integration of vector valued functions**

We define the *integral of a vector valued function* as the integral of each component. This definition holds for both definite and indefinite integrals.

**Example \( \PageIndex{3} \)**

Evaluate

\[
\int (\sin t) \hat{i} + 2t \hat{j} - 8t^3 \hat{k} \; dt.
\]

**Solution**

Just take the integral of each component
Notice that we have introduced three different constants, one for each component.

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