4.1: Differentiation and Integration of Vector Valued Functions

The formal definition of the derivative of a vector valued function is very similar to the definition of the derivative of a real valued function.

Definition: The Derivative of a Vector Valued Function

Let \( r(t) \) be a vector valued function, then

\[
  r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}.
\]

Because the derivative of a sum is the sum of the derivative, we can find the derivative of each of the components of the vector valued function to find its derivative.

Example

\[
  \frac{d}{dt} (3 \hat{i} + \sin t \hat{j}) = \cos t \hat{j}
\]

\[
  \frac{d}{dt} \left(3t^2 \hat{i} + \cos(4t) \hat{j} + te^t \hat{k}\right) = 6t \hat{i} - 4 \sin(t) \hat{j} + (e^t + te^t) \hat{k}
\]

Properties of Vector Valued Functions

All of the properties of differentiation still hold for vector values functions. Moreover because there are a variety of ways of defining multiplication, there is an abundance of product rules.
Suppose that \( \text{v}(t) \) and \( \text{w}(t) \) are vector valued functions, \( f(t) \) is a scalar function, and \( c \) is a real number then

1. \[ \frac{d}{dt} \left( \text{v}(t) + \text{w}(t) \right) = \frac{d}{dt}\text{v}(t) + \frac{d}{dt}\text{w}(t) \],
2. \[ \frac{d}{dt} c\text{v}(t) = c\, \frac{d}{dt} \text{v}(t) \],
3. \[ \frac{d}{dt}(f(t) \text{v}(t)) = f'(t) \text{v}(t) + f(t) \text{v}(t)' \],
4. \[ \left( \text{v}(t) \cdot \text{w}(t) \right)' = \text{v}'(t) \cdot \text{w}(t) + \text{v}(t) \cdot \text{w}'(t) \],
5. \[ ((v(t) \times \text{w}(t))' = \text{v}'(t) \times \text{w}(t) + \text{v}(t) \times \text{w}'(t)) \],
6. \[ \frac{d}{dt} \text{v}(f(t)) = \text{v}(t)'(f(t)) f'(t) \].

Example \( \PageIndex{2} \)

Show that if \( \text{r} \) is a differentiable vector valued function with constant magnitude, then

\[ [ \text{r} \cdot \text{r}' = 0. ] \]

**Solution**

Since \( \text{r} \) has constant magnitude, call its magnitude \( k \),

\[ [ k^2 = |r|^2 = r \cdot r. ] \]

Taking derivatives of the left and right sides gives

\[ [ 0 = (r \cdot r)' = r' \cdot r + r \cdot r' ] \]

\[ [ = r \cdot r' + r \cdot r' = 2r \cdot r'. ] \]

Divide by two and the result follows

**Integration of vector valued functions**

We define the *integral of a vector valued function* as the integral of each component. This definition holds for both definite and indefinite integrals.

Example \( \PageIndex{3} \)

Evaluate

\[ [ \int (\sin t) \hat{\textbf{i}} + 2t\hat{\textbf{j}} - 8t^3 \hat{\textbf{k}} \; dt. ] \]

**Solution**

Just take the integral of each component
\[ \int (\sin t) \, dt \, \hat{\textbf{i}} + \int 2t \, dt \, \hat{\textbf{j}} - \int 8t^3 \, dt \, \hat{\textbf{k}}. \]

\[ = (-\cos t + c_1) \, \hat{\textbf{i}} + (t^2 + c_2) \, \hat{\textbf{j}} + (2t^4 + c_3) \, \hat{\textbf{k}}. \]

Notice that we have introduced three different constants, one for each component.

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