4.5: Path Independence, Conservative Fields, and Potential Functions

For certain vector fields, the amount of work required to move a particle from one point to another is dependent only on its initial and final positions, not on the path it takes. Gravitational and electric fields are examples of such vector fields. This section will discuss the properties of these vector fields.

Definition: Path Independent and Conservative

Let \( \mathbf{F} \) be a vector field defined on an open region \( D \) in space, and suppose that for any two points \( A \) and \( B \) in \( D \) the line integral

\[
\int_{C} \mathbf{F} \cdot \mathbf{d}r
\]

along a path \( C \) from \( A \) to \( B \) in \( D \) is the same over all paths from \( A \) to \( B \). Then the integral \( \int_{C} \mathbf{F} \cdot \mathbf{d}r \) is path independent in \( D \) and the field \( \mathbf{F} \) is conservative on \( D \).

Potential Function

Definition: If \( \mathbf{F} \) is a vector field defined on \( D \) and \( \mathbf{F} = \nabla f \) for some scalar function \( f \) on \( D \), then \( f \) is called a potential function for \( \mathbf{F} \). You can calculate all the line integrals in the domain \( \mathbf{F} \) over any path between \( A \) and \( B \) after finding the potential function \( f \)

\[
\int_{A}^{B} \mathbf{F} \cdot \mathbf{d}r = \int_{A}^{B} \nabla f \cdot \mathbf{d}r = f(B) - f(A)
\]

This can be related back to the Fundamental Theorem of Calculus, since the gradient can be thought of as similar to the
derivative. Another important property of conservative vector fields is that the integral of $F$ around any closed path $D$ is always 0.

Assumptions on Curves, Vector Fields, and Domains

For computational sake, we have to assume the following properties regarding the curves, surfaces, domains, and vector fields:

1. The curves we consider are piecewise smooth, meaning they are composed of many infinitesimally small, smooth pieces connected end to end.
2. We assume that the domain $D$ is a simply connected open region, meaning that any two points in $D$ can be joined by a smooth curve within the region and that every loop in $D$ can be contracted to a point in $D$ without ever leaving $D$.

Theorem 1: Fundamental Theorem of Line Integrals

Let $C$ be a smooth curve joining the point $A$ to point $B$ in the plane or in space and parametrized by $\mathbf{r}(t)$. Let $f$ be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on a domain $D$ containing $C$. Then $
abla f$ on a domain $D$ containing

\[\int_{C} \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)\]

Proof

Suppose that $A$ and $B$ are two points in region $D$ and that the curve $C$ is given by $\mathbf{r}(t)$ is a smooth curve in $D$ that joins points $A$ and $B$. Along $C$, $f$ is a differentiable function of $t$ and

\[\begin{align*} \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \\ &= \nabla f \cdot \left( \frac{\mathrm{d} x}{\mathrm{d} t} \mathbf{i} + \frac{\mathrm{d} y}{\mathrm{d} t} \mathbf{j} + \frac{\mathrm{d} z}{\mathrm{d} t} \mathbf{k} \right) \\ &= \mathbf{F} \cdot \frac{\mathrm{d} \mathbf{r}}{\mathrm{d} t} \end{align*}\]

Therefore,

\[\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{t=a}^{t=b} \mathbf{F} \cdot \frac{\mathrm{d} \mathbf{r}}{\mathrm{d} t} \, dt = \int_{a}^{b} \frac{\mathrm{d} f}{\mathrm{d} t} \, dt \]

*Note:

$\mathbf{r}(a) = A, \quad \mathbf{r}(b) = B$

Which means:

$\left[ f(g(t), h(t), k(t)) \right]_{a}^{b} = f(B) - f(A)$
Thus proving Theorem 1. This shows us that the integral of a gradient field is easy to compute, provided we know the function $f$.

\[ \text{(square)} \]

As mentioned earlier, this is very similar to the Fundamental Theorem of Calculus both in theory and importance. Like the FTC, it provides us with a way to evaluate line integrals without limits of Riemann sums.

**Theorem 2: Conservative Fields are Gradient Fields**

Let $\mathbf{F}=M\hat{i}+N\hat{j}+P\hat{k}$ be a vector field whose components are continuous throughout an open connected region $D$ in space. Then $\mathbf{F}$ is conservative if and only if it is a gradient field $\nabla f$ for a differentiable function $f$.

**Proof**

If $\mathbf{F}$ is a gradient field, then $\mathbf{F}=\nabla f$ for a differentiable function $f$. By Theorem 1, we know that $\int_C \mathbf{F} \cdot d\mathbf{r}=f(B)-f(A)$ and that the value of the line integral depends only on the two endpoints, not on the path. The line integral is said to be independent and $\mathbf{F}$ is a conservative field.

However, suppose $\mathbf{F}$ is a conservative vector field and we want to find some function $f$ on $D$ such that $\nabla f=\mathbf{F}$. First, we must pick a point $A$ in the domain $D$ such that $f(A)=0$. For any other point $B$, we must define $f(B)$ as equal to $\int_C \mathbf{F} \cdot d\mathbf{r}$ where the curve $C$ is any smooth path in $D$ from $A$ to $B$. Because $F$ is conservative, we know that $f(B)$ is not dependant on $C$ and vice versa. In order to show that $\nabla f=\mathbf{F}$, we need to show that

\[
\frac{\partial f}{\partial x}=M, \frac{\partial f}{\partial y}=N, \frac{\partial f}{\partial z}=P.
\]

Suppose $B$ has coordinates $(x,y,z)$ and a nearby point $B_0=(x_0,y,z)$. By definition, then, the value of function $f$ at the nearby point is $f(B)=\int_{C_0} \mathbf{F} \cdot d\mathbf{r}$ where $C_0$ is any path from $A$ to $B$. We can take path $C$ to be the union between path $C_0$ and line segment $L$ from $B$ to $B_0$. Therefore,

\[
f(x,y,z)=\int_{C_0} \mathbf{F} \cdot d\mathbf{r}+\int_L \mathbf{F} \cdot d\mathbf{r}
\]

We can differentiate this integral, arriving at:

\[
\frac{\partial}{\partial x}f(x,y,z)=\frac{\partial}{\partial x}\left( \int_{C_0} \mathbf{F} \cdot d\mathbf{r}+\int_L \mathbf{F} \cdot d\mathbf{r} \right)
\]

Only the last term of the above equation is dependent on $x$, so

\[
\frac{\partial}{\partial x}f(x,y,z)=\frac{\partial}{\partial x}\left( \int_{C_0} \mathbf{F} \cdot d\mathbf{r} \right)
\]

Now, if we parametrize $(L)$ such that
where \((x_0 \leq t \leq x)\) Then,

\[
\frac{\text{d} \mathbf{r}}{\text{d} t} = \mathbf{i}
\]

and

\[
\int_L \mathbf{f} \cdot d\mathbf{r} = \int_{x_0}^x M(t,y,z) \, dt
\]

Substitution gives us

\[
\frac{\partial }{\partial x} f(x,y,z) = \frac{\partial }{\partial x} \int_{x_0}^x M(t,y,z) \, dt = M(x,y,z)
\]

by the FTC. The partial derivatives

\[
\frac{\partial f}{\partial y} = N
\]

and

\[
\frac{\partial f}{\partial z} = P
\]

follow similarly, showing that

\[
\mathbf{F} = \bigtriangledown f
\]

In other words, \(\mathbf{F} = \bigtriangledown f\) is only true when, for any two points \(A\) and \(B\) in the region \(D\), \(\oint_C \mathbf{F} \cdot d\mathbf{r} = 0\) is independent of the path \(C\) that joins the two points in \(D\).

Theorem 3: Looper Property of Conservative Fields

The following statements are equivalent:

- \(\oint_C \mathbf{F} \cdot d\mathbf{r} = 0\) around every loop (closed curve \(C\)) in \(D\).
- The field \(\mathbf{F}\) is conservative on \(D\).

Proof

**Part 1**

We want to show that for any two points \(A\) and \(B\) in \(D\), the integral of

\[
\int_C \mathbf{F} \cdot d\mathbf{r}
\]
has the same value over any two paths \( (C_1) \) & \( (C_2) \) from \( A \) to \( B \).

We reverse the direction of \( (C_2) \) to make the path \( -(C_2) \) from \( B \) to \( A \).

Together, the two curves \( (C_1) \) & \( -(C_2) \) make a closed loop, which we will call \( C \).

If you recall from earlier in this section, the integral over a closed loop for a conservative field is always 0:

\[
\begin{align*}
\int_{C_1}\mathbf{F}\cdot d\mathbf{r}-\int_{C_2}\mathbf{F}\cdot d\mathbf{r}&=\int_{C_1}\mathbf{F}\cdot d\mathbf{r}+\int_{-C_2}\mathbf{F}\cdot d\mathbf{r} \\
&=\int_C\mathbf{F}\cdot d\mathbf{r} \\
&=0
\end{align*}
\]

Therefore, the integrals over \( (C_1) \) & \( -(C_2) \) must be equal.

**Part 2**

We want to show that the integral over \( \mathbf{F}\cdot d\mathbf{r} \) is zero for any closed loop \( C \). We pick two points \( A \) \& \( B \) on \( C \) and use them to break \( C \) into 2 pieces: \( (C_1) \) from \( A \) to \( B \) and \( (C_2) \) from \( B \) back to \( A \).

Therefore:

\[
\begin{align*}
\oint_C\mathbf{F}\cdot d\mathbf{r}&=\int_{C_1}\mathbf{F}\cdot d\mathbf{r}+\int_{C_2}\mathbf{F}\cdot d\mathbf{r} \\
&=\int_A^B\mathbf{F}\cdot d\mathbf{r}-\int_A^B\mathbf{F}\cdot d\mathbf{r} \\
&=0
\end{align*}
\]

\(\square\)

**Finding Potentials for Conservative Fields**

**Component Test for Conservative Fields:** Let \( \mathbf{F}=M(x,y,z)\hat{\textbf{i}} + N(x,y,z) \hat{\textbf{j}}+ P(x,y,z) \hat{\textbf{k}} \) be a field on a connected and simply connected domain whose component functions have continuous first partial derivatives. Then, \( \mathbf{F} \) is conservative if and only if

\[
\frac{\partial P }{\partial x}=\frac{\partial M}{\partial z}\\
\frac{\partial P }{\partial y}=\frac{\partial N}{\partial z}\\
\frac{\partial N }{\partial x}=\frac{\partial M}{\partial y}
\]

*Note: See Example 2

**Definition:** Exact Differential Forms
Any expression
\[ [M(x,y,z)dx+N(x,y,z)dy+P(x,y,z)dz] \]
is a differential form. A differential form is exact on a domain \( D \) in space if
\[ [M, dx+N, dy+P, dz]=d\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=df \]
for some scalar function \( f \) throughout \( D \).

**Component Test for Exactness of** \( (Mdx+Ndy+Pdz) \): The differential form \( (Mdx+Ndy+Pdz) \) is exact on a connected and simply connected domain if and only if
\[ \begin{align*}
\frac{\partial P}{\partial x} &= \frac{\partial M}{\partial z} \\
\frac{\partial P}{\partial y} &= \frac{\partial N}{\partial z} \\
\frac{\partial N}{\partial x} &= \frac{\partial M}{\partial y}
\end{align*} \]
and
\[ \begin{align*}
\frac{\partial P}{\partial z} &= \frac{\partial M}{\partial y} \\
\frac{\partial P}{\partial x} &= \frac{\partial N}{\partial z} \\
\frac{\partial N}{\partial y} &= \frac{\partial M}{\partial z}
\end{align*} \]
Notice, this is the same as saying the field \( \mathbf{F}=M\hat{i}+N\hat{j}+P\hat{k} \) is conservative.

**Example \( \PageIndex{1} \)**

Suppose the force field \( \mathbf{F}=\bigtriangledown f \) is the gradient of the function \( f(x,y,z)=-\frac{1}{x^2+y^2+z^2} \). Find the work done by \( \mathbf{F} \) in moving an object along a smooth curve \( C \) joining \( (1,0,0) \) to \( (0,0,2) \) that does not pass through the origin.

**Solution**
Since we know that this is a conservative field, we can apply Theorem 1, which shows that regardless of the curve \( C \), the work done by \( \mathbf{F} \) will be as follows:
\[ \begin{align*}
\begin{align}
\int_C \mathbf{F} \cdot d\mathbf{r} &= f(0,0,2)-f(1,0,0) \\
&= -\frac{1}{4} - (-1) \\
&= \frac{3}{4}
\end{align}
\end{align} \]

**Example \( \PageIndex{2} \)**

Show that
\[ \mathbf{F}=(e^x \cos y+yz)\hat{i}+(xz-e^x\sin y)\hat{j}+(xy+z)\hat{k} \]
is conservative over its natural domain and find a potential function for it.
Solution

The natural domain of $\mathbf{F}$ is all of space, which is connected and simply connected. Let's define the following:

\[ M = e^x \cos y + yz \nonumber \]
\[ N = xz - e^x \sin y \nonumber \]
\[ P = xy + z \nonumber \]

and calculate

\[ \frac{\partial P}{\partial x} = y = \frac{\partial M}{\partial z} \nonumber \]
\[ \frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z} \nonumber \]
\[ \frac{\partial N}{\partial x} = -e^x \sin y = \frac{\partial M}{\partial y} \nonumber \]

Because the partial derivatives are continuous, $\mathbf{F}$ is conservative. Now that we know there exists a function $f$ where the gradient is equal to $\mathbf{F}$, let's find $f$.

\[ \frac{\partial f}{\partial x} = e^x \cos y + yz \nonumber \]
\[ \frac{\partial f}{\partial y} = xz - e^x \sin y \nonumber \]
\[ \frac{\partial f}{\partial z} = xy + z \nonumber \]

If we integrate the first of the three equations with respect to $x$, we find that

\[ f(x,y,z) = \int (e^x \cos y + yz) dx = e^x \cos y + xyz + g(y,z) \]

where $g(y,z)$ is a constant dependent on $y$ and $z$ variables. We then calculate the partial derivative with respect to $y$ from this equation and match it with the equation of above.

\[ \frac{\partial}{\partial y} (f(x,y,z)) = -e^x \sin y + xz + \frac{\partial g}{\partial y} = xz - e^x \sin y \nonumber \]

This means that the partial derivative of $g$ with respect to $y$ is 0, thus eliminating $g$ from $g$ entirely and leaving it as a function of $z$ alone.

\[ f(x,y,z) = e^x \cos y + xyz + h(z) \nonumber \]

We then repeat the process with the partial derivative with respect to $z$.

\[ \frac{\partial}{\partial z} (f(x,y,z)) = xy + \frac{\partial h}{\partial z} = xy + z \nonumber \]

which means that
\[ \frac{dh}{dz} = z \]
so we can find \( h(z) \) by integrating:

\[ h(z) = \frac{z^2}{2} + C. \]

Therefore,

\[ f(x,y,z) = e^x \cos y + xyz + \frac{z^2}{2} + C. \]

We still have infinitely many potential functions for \( F \)-one at each value of \( C \).

Example \( \PageIndex{3} \)

Show that \( ydx + xdy + 4dz \) is exact and evaluate the integral

\[ \int_{(1,1,1)}^{(2,3,-1)} ydx + xdy + 4dz \]

over any path from \( (1,1,1) \) to \( (2,3,-1) \).

Solution

We let \( M = y \), \( N = x \), and \( P = 4 \). Apply the Test for Exactness:

\[ \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y} \]
\[ \frac{\partial N}{\partial z} = 0 = \frac{\partial P}{\partial y} \]
\[ \frac{\partial N}{\partial z} = 0 = \frac{\partial P}{\partial y}. \]

This proves that \( ydx + xdy + 4dz \) is exact, so

\[ ydx + xdy + 4dz = df \]

for some function \( f \), and the integral's value is \( (f(2,3,-1) - f(1,1,1)) \).

We find \( f \) up to a constant by integrating the following equations:

\[ \frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = 4 \]

From the first equation, we get that \( f(x,y,z) = xy + g(y,z) \)

The second equation tells us that \( \frac{\partial f}{\partial x} = x + \frac{\partial g}{\partial y} \{} \partial y \} = x \)
Therefore,
\[
\frac{\partial g}{\partial y} = 0
\]
Hence,
\[
f(x,y,z) = xy + h(z)
\]
The third equation tells us that
\[
\frac{\partial f}{\partial z} = 0 + \frac{dh}{dz} = 4
\]
so \(h(z) = 4z + C\)
Therefore,
\[
f(x,y,z) = xy + 4z + C
\]
By substitution, we find that:
\[
f(2,3,-1) - f(1,1,1) = 2 + C - (5 + C) = -3
\]

References

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