3.7: Moments and Centers of Mass

This section shows how to calculate the masses and moments of two- and three-dimensional objects in Cartesian \((x,y,z)\) coordinates.

**Mass**

We saw before that the double integral over a region of the constant function 1 measures the area of the region. If the region has uniform density 1, then the mass is the density times the area which equals the area. What if the density is not constant. Suppose that the density is given by the continuous function

\[
\text{Density} = \rho(x,y).
\]

In this case we can cut the region into tiny rectangles where the density is approximately constant. The area of mass rectangle is given by

\[
\text{Mass} = (\text{Density})(\text{Area})
\]

\[
= (\rho(x,y)) (\Delta x \Delta y)
\]

You probably know where this is going. If we add all to masses together and take the limit as the rectangle size goes to zero, we get a double integral.

**Definition: Mass of a Two-Dimensional lamina**

Let \(\rho(x,y)\) be the density of a lamina (flat sheet) \((R)\) at the point \((x,y)\). Then the total mass of the lamina is the double integral
\[ \text{Mass}_{\text{lamina}} = \iint \rho (x,y) \, dy \, dx \label{lamina} \]

or written as an integral over an area \((A)\):

\[ \text{Mass}_{\text{lamina}} = \iint_a^b \rho \, dA \]

Example (PageIndex{1})

A rectangular metal sheet with \(2 < x < 5\) and \(0 < y < 3\) has density function

\[ \rho(x,y) = x + y. \nonumber \]

Set up the double integral that gives the mass of the metal sheet.

Solution

We just have to evaluate the integral in Equation \ref{lamina}

\[ \int_2^5 \int_0^3 (x+y) \, dy \, dx. \nonumber \]

Extending this to three-dimensional solids requires redefining \(\rho(x,y,z)\) to be the density (mass per unit volume) of an object occupying a region \(D\) in space. The integral over \(D\) gives us the mass of the object. To see why, imagine partitioning the object into \(n\) mass elements. And when summing these mass elements up, it is the total mass.

\[
\begin{align*}
M &= \lim_{n \to \infty} \sum_{k=1}^n \Delta m_k \\
&= \lim_{n \to \infty} \sum_{k=1}^n \rho(x_k,y_k,z_k) \Delta V_k \\
&= \iiint_D \rho(x,y,z) \, dV
\end{align*}
\]

The integral of \(\rho(x,y,z)\) gives us the mass of the object.

Definition: Mass of a Three-Dimensional Solid

Let \(\rho(x,y,z)\) be the density of a solid \(R\) at the point \((x,y,z)\). Then the total mass of the solid is the triple integral

\[ \text{Mass}_{\text{solid}} = \iiint \rho (x,y,z) \, dy \, dx \, dz \label{solid} \]

or written as an integral over an volume \((V)\):

\[ \text{Mass}_{\text{solid}} = \iiint_a^b \rho \, dV \]

Moments and Center of Mass

The moments about an axis are defined by the product of the mass times the distance from the axis.

\[ M_x = (\text{Mass}(y)) \]

UC Davis ChemWiki is licensed under a Creative Commons Attribution-Noncommercial-Share Alike 3.0 United States License.
If we have a region $R$ with density function $\rho(x,y)$, then we do the usual thing. We cut the region into small rectangles for which the density is constant and add up the moments of each of these rectangles. Then take the limit as the rectangle size approaches zero. This will give us the total moment.

Definition: Moments of Mass and Center of Mass

Suppose that $\rho(x,y)$ is a continuous density function on a lamina $R$. Then the moments of mass are

$$M_x = \int_0^1 \int_0^2 k(x^2 + y^2) y \, dy \, dx$$

and

$$M_y = \int_0^1 \int_0^2 k(x^2 + y^2) x \, dy \, dx$$

and if $M$ is the mass of the lamina, then the center of mass is

$$\left( \bar{x}, \bar{y} \right) = \left( \frac{M_y}{M}, \frac{M_x}{M} \right).$$

Example \(\PageIndex{2}\)

Set up the integrals that give the center of mass of the rectangle with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ and density function proportional to the square of the distance from the origin. Use a calculator or computer to evaluate these integrals.

**Solution**

The mass is given by

$$M = \int_0^1 \int_0^2 k(x^2 + y^2) \, dy \, dx = \frac{2k}{3}.$$  

The moments are given by (definition 2a):

$$M_x = \int_0^1 \int_0^2 k(x^2 + y^2) y \, dy \, dx$$

and

$$M_y = \int_0^1 \int_0^2 k(x^2 + y^2) x \, dy \, dx.$$  

These evaluate to

$$M_x = \frac{5k}{12}$$

and

$$M_y = \frac{5k}{12}.$$
\[ M_y = \frac{5k}{12} \, \text{nonumber} \]

It should not be a surprise that the moments are equal since there is complete symmetry with respect to \((x)\) and \((y)\). Finally, we divide to get

\[(x, y) = \left( \frac{5}{8}, \frac{5}{8} \right) \, \text{nonumber} \]

This tells us that the metal plate will balance perfectly if we place a pin at \((\frac{5}{8}, \frac{5}{8})\).

---

**Moments of Inertia**

We often call \((M_x)\) and \((M_y)\) the first moments. They have first powers of \((y)\) and \((x)\) in their definitions and help find the center of mass. We define the *moments of inertia* (or second moments) by introducing squares of \((y)\) and \((x)\) in their definitions. The moments of inertia help us find the kinetic energy in rotational motion. Below is the definition.

**Definition: Moments of Inertia**

Suppose that \((\rho(x, y))\) is a continuous density function on a lamina \((R)\). Then the *moments of inertia* are

\[ I_x = \iint_R \rho(x, y) y^2 \, dy\, dx \]
\[ I_y = \iint_R \rho(x, y) x^2 \, dy\, dx \]

**Exercise** \(\PageIndex{1}\)

Find the moments of inertia for the square metal plate in Example \(\PageIndex{2}\).

---

**First Moment**

The first moment of a **3-D solid region** \((D)\) about a coordinate plane is defined as the triple integral over \((D)\) of the distance from a point \((x, y, z))\) in \((D)\) to the plane multiplied by the density of the solid at that point. First moments about the coordinate planes:

\[ M(yz) = \iiint_D \delta x \, dV \]
\[ M(xz) = \iiint_D \delta y \, dV \]
\[ M(xy) = \iiint_D \delta z \, dV \]

The first moment about the \(y\)-axis is the double integral over the region \((R)\) forming the 2-D **plate** of the distance from the axis multiplied by the density.

\[ M(y) = \iint_R \delta x \, dV \]
\[ M(x) = \iiint_a^b \delta y \, dV \]

**Center of Mass**

We defined *center of mass* located in \( \bar{x}, \bar{y}, \bar{z} \). Then it is found from the first moments:

\[
\bar{x} = \frac{M(y)}{M}
\]

\[
\bar{y} = \frac{M(x)}{M}
\]

**Contributors**

- Shengqiao Luo (UCD)