3.9: Substitutions in Multiple Integrals

This section discusses the translation of a graph from the \(xy\) Cartesian plane to the \(uv\) Cartesian plane and defines the Jacobian.

Introduction

As observed in other sections regarding polar coordinates, some integration of functions on the xyz-space are more easily integrated by translating them to different coordinate systems. These substitutions can make the integrand and/or the limits of integration easier to work with, as "U" Substitution did for single integrals. In this section, we will translate functions from the x-y-z Cartesian coordinate plane to the u-v-w Cartesian coordinate plane to make some integrations easier to solve.

One key component of this translation is called the Jacobian determinant, or simply the Jacobian, which measures how much the volume at a certain point changes when being transformed from one coordinate system to another.

It is important to note that although we are changing the coordinate system on which we graph our function, the theory behind multiple integrals has not changed. The limits of integration still create the domain under the curve, and the integration will help us find the volume of the figure created by the original function and the domain.

Theoretical discussion with Descriptive Elaboration

For any given function \(f(x,y)\), we can define x and y as a function of other variables \(g(u,v)\). To do this, we first find \(u\) and \(v\) as a function of \(x\) and \(y\) that will allow for an easier integrand. Then solve for \(x\) and \(y\) in order to translate the function so that \(x=g(u,v)\ \text{and} \ y=h(u,v)\). This translates the area region from R in the x-y plane to D in
the u-v plane.

Remember:

\[ I = \iint_R f(x,y) \ dA \]

So we must find \( dA \):

\( dA \) changes from \( \left( dx \ dy \right) \) to \( \left( J(u,v) \ right) \ dudv \). Each change in u (\( \Delta u \)) and change in v (\( \Delta v \)) create parallelograms that are small areas \( \Delta A \) or \( dA \). We can find the area of each of these parallelograms (P) by taking the cross product of the two vectors that create it (\( \left( \Delta u \ \text{and} \ \Delta v \right) \)).

\[
\text{Area of P} = \begin{vmatrix} \vec{V}_1 \times \vec{V}_2 \end{vmatrix} = J(u,v) 
\]

\[

\begin{align*}
J(u,v) &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix} \\
&= \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \\
&= \frac{\partial (x,y)}{\partial (u,v)}
\end{align*}
\]

\( |J(u,v)| \) represents the area of the parallelogram, and it is the determinant of the Jacobian matrix, shown above. The Jacobian measures how much the transformation is changing from the region \( R \) to the region \( G \). Therefore, the translation of the integration of \( f(x,y) \) is represented by

\[
\iint_R f(x,y) \ dx \ dy = \iint_G f(g(u,v), h(u,v)) | J(u,v) | \ du \ dv .
\]

The same can be applied for triple integrals, where the translations are represented by

\[
[x=g(u,v,w), \ y=h(u,v,w), \ z=k(u,v,w)]
\]

This method for getting the Jacobian is called cofactor expansion.
Although the introduction focused primarily on translating a Cartesian function to a different Cartesian coordinate system, the concept of the Jacobian can also be used to explain how translations into polar coordinates work as well.

For cylindrical coordinates

\[
\begin{align*}
\begin{vmatrix}
\cos \theta & -r \sin \theta & 0 \\
\sin \theta & r \cos \theta & 0 \\
0 & 0 & 1
\end{vmatrix} \\
\text{Therefore:}
\end{align*}
\]

\[
\iiint_D F(x,y,z) \, dx \, dy \, dz = \iiint_G H(r, \theta, z) \, r \, dr \, d\theta \, dz
\]

For spherical coordinates

\[
\begin{align*}
\begin{vmatrix}
\sin \phi & 0 & 0 \\
r \cos \phi & 0 & 0 \\
0 & 0 & 1
\end{vmatrix} \\
\text{Hence, } (dx,dy,dz) \text{ becomes } (rd,rd,d\theta) \text{ in cylindrical coordinates and } (r^2 \sin \phi \, dr \, d\phi \, d\theta) \text{ in spherical coordinates.}
\end{align*}
\]

Example \(\PageIndex{1}\)

Use the following transformation to evaluate the integral.

\[
\begin{align*}
\begin{vmatrix}
\frac{y}{x} & x \\
\text{and} & v = xy
\end{vmatrix} \\
\end{align*}
\]

\[
\begin{align*}
\iiint \text{Where R is bounded by:} \ 1 \le u \le 2 \text{ and } 1 \le v \le 2
\end{align*}
\]

Solution

First find \(x')\) and \(y')\) as functions of \((u')\) and \((v'):\)

\[
(u = \frac{y}{x}) \ \ (v = xy)
\]
\( y = xu \) \( \rightarrow \) \( x = \frac{v}{yu} \)

\( x = \frac{v}{yu} \)

\( (y = \sqrt{\frac{vu}{u}}) \)

\( y = \sqrt{vu} \)

\( x = g(u,v) = \sqrt{\frac{v}{u}} \) \( \text{and} \) \( y = h(u,v) = \sqrt{vu} \)

Using \( y = g(u,v) \) \( \text{and} \) \( y = h(u,v) \), find the integrand in terms of \( u \) \( \text{and} \) \( v \):

\( \frac{y}{x} = \frac{\sqrt{vu}}{\sqrt{\frac{v}{u}}} = u \)

And \( \text{d}A \) changes from \( \text{d}x \text{d}y \) to \( \text{d}u \text{d}v \). The Jacobian equals:

\[
J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}
\]

\[
\frac{\partial x}{\partial u} = \frac{1}{2} u^{-\frac{3}{2}} v^{\frac{1}{2}} \quad \frac{\partial x}{\partial v} = \frac{1}{2} u^{-\frac{1}{2}} v^{-\frac{1}{2}} \\
\frac{\partial y}{\partial u} = \frac{1}{2} u^{-\frac{1}{2}} v^{\frac{1}{2}} \quad \frac{\partial y}{\partial v} = \frac{1}{2} u^{\frac{1}{2}} v^{-\frac{1}{2}}
\]

\[
J(u,v) = \begin{vmatrix} \frac{1}{2} u^{-\frac{3}{2}} v^{\frac{1}{2}} & \frac{1}{2} u^{-\frac{1}{2}} v^{-\frac{1}{2}} \\ \frac{1}{2} u^{-\frac{1}{2}} v^{\frac{1}{2}} & \frac{1}{2} u^{\frac{1}{2}} v^{-\frac{1}{2}} \end{vmatrix} = \left( -\frac{1}{4} u^{-1} -\frac{1}{4} u^{-1} \right) = \frac{1}{2u}
\]

Therefore, evaluate:

\[
\begin{align*}
\int_{1}^{2} \int_{1}^{2} u \left( \frac{1}{2u} \right) \ du \ dv &= \int_{1}^{2} \left. \frac{1}{2} u \right|_{1}^{2} \ dv \ &= \left. \frac{1}{2} v \right|_{1}^{2} \ &= \frac{1}{2}
\end{align*}
\]

Example \( \PageIndex{2} \)

Use the following transformation to evaluate the integral.

\( u = x - \frac{1}{2} y \) \( \text{and} \) \( v = y \)

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\[
\int_0^\frac{1}{2} \int_{\frac{y}{2}}^{\frac{y+4}{2}} y^3 (2x-y) e^{(2x-y)^2} \, dx \, dy
\]

**Solution**

First solve for \( x \) and \( y \):

\[
\begin{align*}
&u = x - \frac{1}{2} y \quad (v = y) \\
&u = x - \frac{1}{2} v \quad (y = v) \\
&x = u + \frac{1}{2} v.
\end{align*}
\]

Then substitute \( x \) and \( y \) for \( g(u,v) \) and \( h(u,v) \) :

The integrand:

\[
\begin{align*}
&y^3 (2x-y) e^{(2x-y)^2} \rightarrow v^3 \left[2 \left(u + \frac{1}{2} v\right) - v\right] e^{\left[2 \left(u + \frac{1}{2} v\right) - v\right]^2} \\
&= v^3 (2u) e^{\left[2u\right]^2} \\
&= (2uv^3) e^{4u^2}.
\end{align*}
\]

The transformation also changes the bounds of integration

\[
\begin{align*}
&x = \frac{y+4}{2} \quad \rightarrow \quad u + \frac{1}{2} v = \frac{v + 4}{2} \\
&y = \frac{y}{2} \quad \rightarrow \quad u + \frac{v}{2} = \frac{v}{2} \\
&u = \frac{4}{2} \\
&u = 0 \\
&u = 2.
\end{align*}
\]

And \( dx \, dy \) changes to \( \left| J(u,v) \right| \, du \, dv \). The Jacobian equals:

\[
\begin{align*}
\begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix}
&= \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \\
&= \begin{vmatrix} 1 & 5 \\
0 & 1
\end{vmatrix}
&= 1.
\end{align*}
\]

Thus,

\[
\begin{align*}
\int_0^2 \int_0^2 2uv^3 e^{4u^2} \, du \, dv = \int_0^2 \int_0^2 8ue^{4u^2} \, du \, dv &= \int_0^2 \int_0^2 e^{4u^2} \, du \, dv = e-1.
\end{align*}
\]

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Example \(\PageIndex{3}\)

Find the mass of an object bounded by

\[(1 \le x \le 2, \ \ \ \ 0 \le xy \le 1, \ \ \ \ 0 \le z \le 2)\]

with a density that can be described by the formula \((x^2 y + 2xyz)\) by using the transformation \((u = x, \ \ \ \ v=xy, \ \ \ \ w = 3z)\).

**Solution**

Set up the integral in cartesian coordinates:

\[
\int_1^2 \int_0^{\dfrac{1}{x}} \int_0^2 x^2y + 2xyz \ dzdydx.
\]

To apply the substitution, first solve for \((x)\) and \((y)\) using the given transformations:

\[
\begin{align*}
\quad &u=x \quad v=xy \quad w=3z \\
\quad &x=u \quad y= \dfrac{v}{x} \quad z = \dfrac{w}{3} \\
\quad &y = \dfrac{v}{u}.
\end{align*}
\]

Then make the appropriate substitutions within the integrand:

\[
x^2 y + 2xyz \rightarrow u^2 \left( \dfrac{v}{u} \right) + 2u \left( \dfrac{v}{u} \right) \left( \dfrac{w}{3} \right) \rightarrow uv + \dfrac{2vw}{3}.
\]

Next, find the new boundaries to the region we want to integrate:

\[
\begin{align*}
\quad &1 \le x \le 2 \quad \rightarrow \quad 1 \le u \le 2 \\
\quad &0 \le xy \le 1 \quad \rightarrow \quad 0 \le v \le 1 \\
\quad &0 \le z \le 2 \quad \rightarrow \quad 0 \le \dfrac{w}{3} \le 2 \quad \rightarrow \quad 0 \le w \le 6.
\end{align*}
\]

To complete the transformation, find the Jacobian:

\[
\begin{align*}
\begin{vmatrix} 1 & 0 & 0 \\
-\dfrac{v}{u^2} & \dfrac{1}{u} & 0 \\
0 & 0 & \dfrac{1}{3}
\end{vmatrix} & = \dfrac{1}{3u}.
\end{align*}
\]

Notice the Jacobian of a lower triangular matrix (the values above the diagonal are all zero) is the multiplication of the diagonal entries. You can confirm this with cofactor expansion.
Using all of our calculated transformations, we can compute the new integral:

\[
\begin{align*}
\text{Mass } &= \int_0^6 \int_0^1 \int_1^2 \left( uv + \frac{2vw}{3} \right) \frac{1}{3u} \ dudvdw \ \& = \\
& = \frac{1}{3} \int_0^6 \int_0^1 \int_1^2 v + \frac{2vw}{3u} \ dudvdw \ \& = \frac{1}{3} \int_0^6 \int_0^1 \left. vu + \frac{2vw}{3u} \ln|u| \right|_1^2 \ dvdw \ \& = \frac{1}{3} \int_0^6 \int_0^1 \left. vu + \frac{2vw}{3} \ln2 \right|_1^2 \ dvdw \ \& = \\
& = \frac{1}{3} \int_0^6 \left. \frac{v^2}{2} + \frac{2w\ln2}{3} \left( \frac{v^2}{2} \right) \right|_0^1 \ dw \ \& = \\
& = \frac{1}{3} \left[ \frac{1}{2} w + \frac{w^2 \ln2}{6} \right]_0^6 \ \& = 1 + 2\ln2 .
\end{align*}
\]

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