For this topic, we will be learning how to calculate the length of a curve in space. The ideas behind this topic are very similar to calculating arc length for a curve in with $x$ and $y$ components, but now, we are considering a third component, $z$.

**Introduction**

The best way to visualize the arc length of a curve is to think of the curve being a piece of string, and then taking that string and extending it out until it becomes a line, which you can then take and measure along a ruler to find the curve's arc length. Since we can’t actually take the curve and measure it that way, we must use another method. We already know how to find the arc length of a curve

\[ r(t)=x(t)\hat{i}+y(t)\hat{j} \]

in a XY-plane for $a \leq t \leq b$. The formula is given as

\[ L=\int_{a}^{b}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} \ dt. \]

Now, we are going to learn how to calculate arc length for a curve in space rather than in just a plane.

![Illustration of a curve getting rectified in order to find its arc length. When rectified, the curve](image-url)
gives a straight line with the same length as the curve's arc length. Image used with permission (Public Domain; Lucas V. Barbosa).

**Arc Length Along A Space Curve**

Calculating the arc length for a curve in space is very similar to calculating the arc length for a curve in the plane. All we need to do is add a $z$ term to the formula for the arc length of a plane curve. So the length of a parameterized curve in space

$$r(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k}$$

from $a \leq t \leq b$ is

$$L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt.$$  

From there, we see that

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

is just the magnitude of $v(t)$, the length of the velocity vector $\frac{dr}{dt}$. So we can rewrite the arc length formula.

$$L = \int_{a}^{b} \sqrt{|v|} \, dt.$$  

Another form of this equation that should look familiar is

$$s(t) = \int_{t_0}^{t} \sqrt{|v(\tau)|} \, d\tau . \quad \label{eq5}$$

This equation was used for curves in planes and still applies to space curves. We use $s(t)$ as the arc length parameter for a space curve, to help represent the properties of a space curve's shape. $\tau$ is used because we use $t$ in the bounds of integration.

**Example \(\PageIndex{1}\): An Example of Arc Length**

While in math class, a student tosses a paper airplane. The paper airplane's flight lasts for 2.5 seconds and can be approximated by $r(t) = 1.2 \sin t \hat{i} + 1.2 \cos t \hat{j} + 0.4 t \hat{k}$. What is the total distance that the airplane travels? Units are in feet.

**Solution**

Using one of the equations for arc length above, we can calculate the length of the paper airplane's path:

$$L = \int_{a}^{b} \sqrt{|v|} \, dt$$

$$v = \frac{dr}{dt} = 1.2 \cos t \hat{i} - 1.2 \sin t \hat{j} + 0.4 \hat{k}.$$
\[
L = \int_{0}^{2.5} \sqrt{(1.2 \cos t)^2 + (-1.2 \sin t)^2 + (.4)^2} \, dt
= \int_{0}^{2.5} \sqrt{1.6} \, dt = 3.162 \text{ feet}
\]

**Unit Tangent Vector**

We can manipulate Equation \ref{eq5} to give us the relationship

\[
\frac{ds}{dt} = |v(t)|
\]

which simply says the rate at which something moves along the path of the curve is equal to its speed.

Now, \(\frac{dr}{dt}\) or \(|v|\) is always tangent to the curve \((r(t))\) and we can calculate the a unit vector \(T\) that is also tangent to \((r(t))\) from what we already know.

\[
T = \frac{v}{|v|}.
\]

We can rewrite this to give us:

\[
\frac{dt}{ds} = \frac{1}{|v|},
\]

\[
\frac{dr}{dt} \frac{dt}{ds} = v \frac{1}{|v|} = \frac{v}{|v|} = T = \frac{dr}{ds}.
\]

**Example** \(\PageIndex{2}\): Finding the Unit Tangent Vector of a Curve

Find the unit tangent vector of the curve \(r(t)=e^{t^3} \hat{i} + \ln(t+5) \hat{j} + 5 \cos t \hat{k}\)

**Solution**

We know that

\[
T = \frac{v}{|v|}.
\]

So first we find \(|v|\)

\[
v = \frac{dr}{dt}.
\]

\[
\frac{dr}{dt} = 3x^2e^{x^3} \hat{i} + \frac{1}{x+5} \hat{j} - 5 \sin t \hat{k}.
\]

Then we find \(|v|\)

\[
|v| = \frac{ds}{dt}.
\]

\[
\frac{ds}{dt} = \sqrt{9x^4e^{2x^3} + \frac{1}{(x+5)^2} + 25 \sin^2 t}.
\]

Now we can find \(|T|\)
\[ T = \frac{3x^2e^{x^3}\hat{i} + \frac{1}{x+5}\hat{j} - 5\sin t\hat{k}}{\sqrt{9x^4e^{2x^3} + \frac{1}{(x+5)^2} + 25\sin^2 t}}. \]

**Contributors**

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