2.6: Tangential and Normal Components of Acceleration

This section breaks down acceleration into two components called the tangential and normal components. Similar to how we break down all vectors into \( \hat{i} \), \( \hat{j} \), and \( \hat{k} \) components, we can do the same with acceleration. The addition of these two components will give us the overall acceleration.

Introduction

We're used to thinking about acceleration as the second derivative of position, and while that is one way to look at the overall acceleration, we can further break down acceleration into two components: tangential and normal acceleration. The tangential acceleration, denoted \( a_T \), allows us to know how much of the acceleration acts in the direction of motion. The normal acceleration \( a_N \) is how much of the acceleration is orthogonal to the tangential acceleration.

Remember that vectors have magnitude AND direction. The tangential acceleration is a measure of the rate of change in the magnitude of the velocity vector, i.e. speed, and the normal acceleration are a measure of the rate of change of the direction of the velocity vector.

This approach to acceleration is particularly useful in physics applications, because we need to know how much of the total acceleration acts in any given direction. Think for example of designing brakes for a car or the engine of a rocket. Why might it be useful to separate acceleration into components?

Theoretical discussion with descriptive elaboration

We can find the tangential acceleration by using Chain Rule to rewrite the velocity vector as follows:
Now, since acceleration is simply the derivative of velocity, we find that:

\[
\begin{align}
\mathbf{a} &= \dfrac{\mathrm{d\mathbf{v}}}{\mathrm{d} t} = \dfrac{\mathrm{d}^2\mathbf{s}}{\mathrm{d} t^2}\mathbf{T} + \dfrac{\mathrm{d}s}{\mathrm{d} t}\left(\dfrac{\mathrm{d}\mathbf{T}}{\mathrm{d} s}\dfrac{\mathrm{d} s}{\mathrm{d} t}\right) \\
&= \dfrac{\mathrm{d}^2\mathbf{s}}{\mathrm{d} t^2}\mathbf{T} + \kappa \left(\dfrac{\mathrm{d} s}{\mathrm{d} t}\right)^2\mathbf{N}
\end{align}
\]

Note

\[
\dfrac{\mathrm{d}\mathbf{T}}{\mathrm{d} s} = \kappa \mathbf{N}
\]

This, in turn, gives us the definition for acceleration by components.

Definition: acceleration vector

If the acceleration vector is written as

\[
\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N},
\]

then

\[
a_T = \dfrac{\mathrm{d}^2\mathbf{s}}{\mathrm{d} t^2} = \dfrac{\mathrm{d}}{\mathrm{d} t}|v| \quad \text{and} \quad a_N = \kappa \left(\dfrac{\mathrm{d} s}{\mathrm{d} t}\right)^2 = \kappa |v|^2
\]

To calculate the normal component of the acceleration, use the following formula:

\[
a_N = \sqrt{|a|^2 - a_T^2} \label{Normal}
\]

We can relate this back to a common physics principal—uniform circular motion. In uniform circulation motion, when the speed is not changing, there is no tangential acceleration, only normal acceleration pointing towards the center of circle. Why do you think this is? Hint: look in the introduction section for the difference between the two components of acceleration.

Example \(\PageIndex{1}\)

Without finding \( \mathbf{T} \) and \( \mathbf{N} \), write the acceleration of the motion

\[
\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}
\]

for \( t > 0 \).
To solve this problem, we must first find the particle’s velocity.

\[
\mathbf{v} = \frac{\mathrm{d} \mathbf{r}}{\mathrm{d} t} = (-\sin t + t\cos t) \hat{i} + (\cos t - t\sin t) \hat{j} = (t\cos t) \hat{i} + (t\sin t) \hat{j}
\]

Next find the speed.

\[
|v| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t|
\]

When \((t>0),\) \(|t|\) simply becomes \(t\).

We know that \(a_T = \frac{\mathrm{d}}{\mathrm{d} t}|v|\), which we can use to find that \(a_T = 1\).

Now that we know \(a_T\), we can use it to find \(a_N\) using Equation \(\ref{Normal}\).

\[
|\mathbf{a}|^2 = t^2 + 1
\]

\[
a_N = \sqrt{(t^2 + 1) - 1} = t
\]

By substituting this back into the original definition, we find that

\[
|\mathbf{a}| = (1)\mathbf{T} + (t)\mathbf{N} = \mathbf{T} + t\mathbf{N}
\]

Example \(\PageIndex{2}\)

Write \(\mathbf{a}\) in the form \(\mathbf{a}=a_T\mathbf{T}+a_N\mathbf{N}\) without finding \(\mathbf{T}\) or \(\mathbf{N}\).

\[
\mathbf{r}(t) = (t+1) \hat{i} + 2t \hat{j} + t^2 \hat{k}
\]

\[
\mathbf{v} = t \hat{i} + 2 \hat{j} + 2t \hat{k}
\]

\[
|\mathbf{v}| = \sqrt{5 + 4t^2}
\]

\[
a_T = \frac{1}{2}(5 + 4t^2)^{-\frac{1}{2}}(8t) = 4t(5 + 4t^2)^{-\frac{1}{2}}
\]

\[
a_T(1) = \frac{4}{3}
\]

\[
\mathbf{a} = 2 \hat{k}
\]

\[
\mathbf{a}(1) = 2 \hat{k}
\]

\[
\mathbf{a}(t) = 2 \hat{k}
\]
Now we use Equation \ref{Normal}:

\[
\begin{align}
  a_N &= \sqrt{|a|^2 - a_T^2} \\
  &= \sqrt{2^2 - \left( \frac{4}{3} \right)^2} \\
  &= \sqrt{\frac{20}{9}} \\
  &= \frac{2\sqrt{5}}{3}
\end{align}
\]

\[
\mathbf{a}(1) = \frac{4}{3}\mathbf{T} + \frac{2\sqrt{5}}{3}\mathbf{N}
\]

References


Contributors and Attributions

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