### 1.7: Tangent Planes and Normal Lines

#### Tangent Planes

Let \( z = f(x,y) \) be a function of two variables. We can define a new function \( F(x,y,z) \) of three variables by subtracting \( z \). This has the condition

\[
F(x,y,z) = 0.
\]

Now consider any curve defined parametrically by

\[
x = x(t), \quad y = y(t), \quad z = z(t).
\]

We can write,

\[
F(x(t), y(t), z(t)) = 0.
\]

Differentiating both sides with respect to \( t \), and using the chain rule gives

\[
F_x(x, y, z) x' + F_y(x, y, z) y' + F_z(x, y, z) z' = 0.
\]

Notice that this is the dot product of the gradient function and the vector \( \langle x', y', z' \rangle \),

\[
\nabla F \cdot \langle x', y', z' \rangle = 0.
\]

In particular the gradient vector is orthogonal to the tangent line of any curve on the surface. This leads to:
Definition: Tangent Plane

Let $F(x, y, z)$ define a surface that is differentiable at a point $(x_0, y_0, z_0)$, then the tangent plane to $F(x, y, z)$ at $(x_0, y_0, z_0)$ is the plane with normal vector

$$\nabla F(x_0, y_0, z_0)$$

that passes through the point $(x_0, y_0, z_0)$. In particular, the equation of the tangent plane is

$$\nabla F(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.$$ 

Example \(\PageIndex{1}\)

Find the equation of the tangent plane to

$$z = 3x^2 - xy$$

at the point $(1, 2, 1)$.

Solution

We let

$$F(x, y, z) = 3x^2 - xy - z$$

then

$$\nabla F = \langle 6x - y, -x, -1 \rangle.$$ 

At the point $(1, 2, 1)$, the normal vector is

$$\nabla F(1, 2, 1) = \langle 4, -1, -1 \rangle.$$ 

Now use the point normal formula for a plane

$$\langle 4, -1, -1 \rangle \cdot \langle x - 1, y - 2, z - 1 \rangle = 0,$$

or

$$4(x - 1) - (y - 2) - (z - 1) = 0.$$ 

Finally we get

$$4x - y - z = 1.$$
**Normal Lines**

Given a vector and a point, there is a unique line parallel to that vector that passes through the point. In the context of surfaces, we have the gradient vector of the surface at a given point. This leads to the following definition.

**Definition: Normal Line**

Let \( F(x,y,z) \) define a surface that is differentiable at a point \((x_0,y_0,z_0)\), then the **normal line to** \( F(x,y,z) \) at \((x_0,y_0,z_0)\) is the line with normal vector

\[
\nabla F(x_0,y_0,z_0).
\]

that passes through the point \((x_0,y_0,z_0)\). In particular, the equation of the normal line is

\[
\begin{align*}
\dot{x}(t) &= x_0 + F_x(x_0,y_0,z_0) t, \\
\dot{y}(t) &= y_0 + F_y(x_0,y_0,z_0) t, \\
\dot{z}(t) &= z_0 + F_z(x_0,y_0,z_0) t.
\end{align*}
\]

Example \(\PageIndex{2}\)

Find the parametric equations for the normal line to

\[
[x^2yz - y + z - 7 = 0]
\]

at the point \((1,2,3)\).
Solution

We compute the gradient:

\[ \nabla F = \langle 2xyz, x^2z - 1, x^2y + 1 \rangle = \langle 12, 2, 3 \rangle. \]

Now use the formula to find

\[ x(t) = 1 + 12t, \quad y(t) = 2 + 2t, \quad z(t) = 3 + 3t. \]

The diagram below displays the surface and the normal line.

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**Angle of Inclination**

Given a plane with normal vector \( \mathbf{n} \) the *angle of inclination*, \( q \), is defined by

\[ \cos q = \frac{\mathbf{n} \cdot \mathbf{k}}{||\mathbf{n}||}. \]

More generally, if \( F(x,y,z) = 0 \) is a surface, then the angle of inclination at the point \( (x_0,y_0,z_0) \) is defined by the angle of inclination of the tangent plane at the point with

\[ \cos \, q = \frac{\nabla F(x_0, y_0, z_0) \cdot \mathbf{k}}{||\nabla F(x_0, y_0, z_0)||}. \]

Example \( \PageIndex{3} \)

Find the angle of inclination of
\[
\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{8} = 1
\]

at the point \((1,1,2)\).

**Solution**

First compute

\[\nabla F = \langle \frac{x}{2}, \frac{y}{2}, \frac{z}{4} \rangle.\]

Now plug in to get

\[\nabla F(1,1,2) = \langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rangle.\]

We have

\[|\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rangle \cdot \hat{k}| = \frac{1}{2}.\]

Also,

\[||\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rangle|| = \frac{\sqrt{3}}{2}.\]

Hence

\[\cos q = \frac{\frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{3}}.\]

So the angle of inclination is

\[q = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 0.955 \text{ radians}.\]

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**The Tangent Line to a Curve**

**Example** \((\PageIndex{4})\)

Find the tangent line to the curve of intersection of the sphere

\[x^2 + y^2 + z^2 = 30\]

and the paraboloid

\[z = x^2 + y^2\]

at the point \((1,2,5)\).

**Solution**
We find the gradient of the two surfaces at the point

\[ \nabla(x^2 + y^2 + z^2) = \langle 2x, 2y, 2z \rangle = \langle 2, 4, 10 \rangle \]

and

\[ \nabla(x^2 + y^2 - z) = \langle 2x, 2y, -1 \rangle = \langle 2, 4, -1 \rangle. \]

These two vectors will both be perpendicular to the tangent line to the curve at the point, hence their cross product will be parallel to this tangent line. We compute

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
2 & 4 & 10 \\
2 & 4 & -1 
\end{vmatrix}
= -44 \hat{i} + 22 \hat{j}.
\]

Hence the equation of the tangent line is

\[ \begin{align*}
x(t) &= 1 - 44t \\
y(t) &= 2 + 22t \\
z(t) &= 5.
\end{align*} \]

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