11.1: Riemann integral over Rectangles

Riemann integral over rectangles

Note: FIXME1 lectures

As in chapter FIXME, we define the Riemann integral using the Darboux upper and lower integrals. The ideas in this section are very similar to integration in one dimension. The complication is mostly notational.

Rectangles and partitions

Let \((a^1,a^2,\ldots,a^n)\) and \((b^1,b^2,\ldots,b^n)\) be such that \((a^k \leq b^k)\) for all \((k)\). A set of the form \([[a^1,b^1) \times [a^2,b^2) \times \cdots \times [a^n,b^n)]]\) is called a closed rectangle. If \((a^k < b^k)\), then a set of the form \(((a^1,b^1) \times (a^2,b^2) \times \cdots \times (a^n,b^n))\) is called an open rectangle.

For an open or closed rectangle \(R := [a^1,b^1] \times [a^2,b^2] \times \cdots \times [a^n,b^n] \subset \mathbb{R}^n\) or \(R := (a^1,b^1) \times (a^2,b^2) \times \cdots \times (a^n,b^n) \subset \mathbb{R}^n\), we define the \(n\)-dimensional volume by \(V(R) := (b^1-a^1) (b^2-a^2) \cdots (b^n-a^n)\).

A partition \(P\) of the closed rectangle \(R = [a^1,b^1] \times [a^2,b^2] \times \cdots \times [a^n,b^n]\) is a finite set of partitions \(P^1,P^2,\ldots,P^n\) of the intervals \([a^1,b^1], [a^2,b^2],\ldots,[a^n,b^n]\). That is, for every \((k)\) there is an integer \(\ell_k\) and the finite set of numbers \(P^k = \{ x_{0,k}, x_{1,k}, x_{2,k}, \ldots, x_{\ell_k,k} \}\) such that \((a^k \leq x_{0,k} < x_{1,k} < x_{2,k} < \cdots < x_{\ell_k,k} \leq b^k)\). Picking a set of \((n)\) integers \(\{j_1,j_2,\ldots,j_n\}\) where \(j_k \in \{1,2,\ldots,\ell_k\}\) we get the subrectangle \([x_{j_1-1}^1, x_{j_1}^1] \times [x_{j_2-1}^2, x_{j_2}^2] \times \cdots \times [x_{j_n-1}^n, x_{j_n}^n]\) of \(R\).
For simplicity, we order the subrectangles somehow and we say \(\{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_N\}\) are the subrectangles corresponding to the partition \(\mathcal{P}\) of \(\mathcal{R}\). In other words we subdivide the original rectangle into many smaller subrectangles. It is not difficult to see that these subrectangles cover our original \(\mathcal{R}\), and their volume sums to that of \(\mathcal{R}\). That is \(\mathcal{R} = \bigcup_{j=1}^{N} \mathcal{R}_j\) and \(V(\mathcal{R}) = \sum_{j=1}^{N} V(\mathcal{R}_j)\).

When \(\mathcal{R}_k = [x_{j_1-1}^{1}, x_{j_1}^{1}] \times [x_{j_2-1}^{2}, x_{j_2}^{2}] \times \cdots \times [x_{j_n-1}^{n}, x_{j_n}^{n}]\) then \(V(\mathcal{R}_k) = \Delta x_{j_1}^{1} \Delta x_{j_2}^{2} \cdots \Delta x_{j_n}^{n} = (x_{j_1}^{1} - x_{j_1-1}^{1}) (x_{j_2}^{2} - x_{j_2-1}^{2}) \cdots (x_{j_n}^{n} - x_{j_n-1}^{n})\).

Let \(\mathcal{R} = \mathbb{R}^n\) be a closed rectangle and let \(f : \mathcal{R} \to \mathbb{R}\) be a bounded function. Let \(\mathcal{P}\) be a partition of \([a, b]\). Let \(\mathcal{R}_i\) be a subrectangle corresponding to \(\mathcal{P}\) that has \(N\) subrectangles. Define

\[
\begin{aligned}
& m_i := \inf \{ f(x) : x \in \mathcal{R}_i \} , \\
& M_i := \sup \{ f(x) : x \in \mathcal{R}_i \} , \\
& L(\mathcal{P}, f) := \sum_{i=1}^{N} m_i V(\mathcal{R}_i) , \\
& U(\mathcal{P}, f) := \sum_{i=1}^{N} M_i V(\mathcal{R}_i) .
\end{aligned}
\]

We start proving facts about the Darboux sums analogous to the one-variable results.

Upper and lower integrals

By the set of upper and lower Darboux sums are bounded sets and we can take their infima and suprema. As before, we now make the following definition.

If \(f : \mathcal{R} \to \mathbb{R}\) is a bounded function on a closed rectangle \(\mathcal{R}\). Define

\[
\begin{aligned}
& \underline{\int}_{\mathcal{R}} f := \sup \{ \text{a partition of } \mathcal{R} \} \sum_{i=1}^{N} m_i V(\mathcal{R}_i) , \\
& \overline{\int}_{\mathcal{R}} f := \inf \{ \text{a partition of } \mathcal{R} \} \sum_{i=1}^{N} M_i V(\mathcal{R}_i) .
\end{aligned}
\]

As in one dimension we have refinements of partitions.

Let \(\mathcal{R} = \mathbb{R}^n\) be a closed rectangle and let \(\mathcal{P} = \{ P^1, P^2, \ldots, \mathcal{P}^n \}\) be partitions of \(\mathcal{R}\). We say \(\mathcal{P}\) a refinement of \(\mathcal{P}\) if as sets \(\mathcal{P}^k \subset \mathcal{I}\) for all \(k = 1, 2, \ldots, n\).
It is not difficult to see that if \( \tilde{P} \) is a refinement of \( P \), then subrectangles of \( P \) are unions of subrectangles of \( \tilde{P} \). Simply put, in a refinement we took the subrectangles of \( P \) and we cut them into smaller subrectangles.

Suppose \( R \subset \mathbb{R}^n \) is a closed rectangle, \( P \) is a partition of \( R \) and \( \tilde{P} \) is a refinement of \( P \). If \( f \) is a bounded function, then \( L(P, f) \leq L(\tilde{P}, f) \leq U(\tilde{P}, f) \leq U(P, f) \).

Let \( (R_1, R_2, \ldots, R_N) \) be the subrectangles of \( P \) and \( (\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_M) \) be the subrectangles of \( \tilde{P} \). Let \( I_k \) be the set of indices \( j \) such that \( \tilde{R}_j \subset R_k \). We notice that \( R_k = \bigcup_{j \in I_k} \tilde{R}_j \), \( V(R_k) = \sum_{j \in I_k} V(\tilde{R}_j) \), \( L(P, f) = \sum_{k=1}^N m_k V(R_k) = \sum_{k=1}^N \sum_{j \in I_k} m_k V(\tilde{R}_j) = \sum_{j=1}^M \tilde{m}_j V(\tilde{R}_j) = L(\tilde{P}, f) \).

The key point of this next proposition is that the lower Darboux integral is less than or equal to the upper Darboux integral.

Let \( R_0 \subset \mathbb{R}^n \) be a closed rectangle and \( f \) a bounded function. Let \( m, M \in \mathbb{R} \) be such that for all \( x \in R_0 \) we have \( m \leq f(x) \leq M \). Then \( m V(R) \leq \underline{\int_R} f \leq \overline{\int_R} f \leq M V(R) \).

The key of course is the middle inequality in \( \underline{\int_a^b} f(x) \, dx = \overline{\int_a^b} f(x) \, dx \). Let \( \{ P_1, P_2, \ldots, P_n \} \) be partitions of \( R \). Define \( \{ \tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n \} \) by letting \( \tilde{P}_k = P_1 \cup P_2 \cup \cdots \cup P_n \). Then \( \{ \tilde{P}_k \} \) is a partition of \( \{ P_1, P_2, \ldots, P_n \} \) as easily can be checked, and \( \{ \tilde{P}_k \} \) is a refinement of \( \{ P_1, P_2, \ldots, P_n \} \) and a refinement of \( \{ P_1, P_2, \ldots, P_n \} \). By \( \{ L(P_1, f), L(P_2, f), \ldots, L(P_n, f) \} \subset \{ U(P_1, f), U(P_2, f), \ldots, U(P_n, f) \} \), we have \( \{ L(P_1, f), L(P_2, f), \ldots, L(P_n, f) \} \subset \{ U(P_1, f), U(P_2, f), \ldots, U(P_n, f) \} \). Therefore, \( \{ L(P_1, f), L(P_2, f), \ldots, L(P_n, f) \} \subset \{ U(P_1, f), U(P_2, f), \ldots, U(P_n, f) \} \). In other words, for two arbitrary partitions \( \{ P_1, P_2, \ldots, P_n \} \) and \( \{ P_2, \ldots, P_n \} \) we have \( \{ L(P_1, f), L(P_2, f), \ldots, L(P_n, f) \} \subset \{ U(P_1, f), U(P_2, f), \ldots, U(P_n, f) \} \).

The key point of the next proposition is that the lower Darboux integral is less than or equal to the upper Darboux integral.

The Riemann integral

We now have all we need to define the Riemann integral in \( \mathbb{R}^n \) over rectangles. Again, the Riemann integral is only defined on a certain class of functions, called the Riemann integrable functions.

Let \( \{ R \} = \{ R_1, R_2, \ldots, R_n \} \) be a closed rectangle. Let \( f \) be a bounded function such that \( \left\lfloor \int_a^b f(x) \, dx \right\rfloor = \overline{\int_a^b f(x) \, dx} \). Then \( f \) is said to be Riemann integrable. The set of Riemann integrable functions on \( \mathbb{R}^n \) is denoted by \( \{ \text{mathecal} \} \). When \( f \in \{ \text{mathecal} \} \) we define the Riemann integral \( \int f := \underline{\int} f = \overline{\int} f \).
When the variable \(x \in \mathbb{R}^n\) needs to be emphasized we write \(\int_R f(x) ~dx\), \(\int_R f(x^1, \ldots, x^n) ~dx^1 \cdots dx^n\), or \(\int_R f(x) ~dV\).

implies immediately the following proposition.

\[\text{mv:intbound:prop}\]

Let \(f \colon R \to \mathbb{R}\) be a Riemann integrable function on a closed rectangle \(\{x \in \mathbb{R}^n\}\). Let \(m, M \in \mathbb{R}\) be such that \(m \leq f(x) \leq M\) for all \(x \in R\). Then \(m \cdot \text{Vol}(R) \leq \int_R f \leq M \cdot \text{Vol}(R)\).

A constant function is Riemann integrable. Suppose \(f(x) = c\) for all \(x\) on \(R\). Then \(c \cdot \text{Vol}(R) \leq \underline{\int_R} f \leq \overline{\int_R} f \leq c \cdot \text{Vol}(R)\). So \(f\) is integrable, and furthermore \(\int_R f = c \cdot \text{Vol}(R)\).

The proofs of linearity and monotonicity are almost completely identical as the proofs from one variable. We therefore leave it as an exercise to prove the next two propositions. (FIXME add the exercise).

Let \(R \subset \mathbb{R}^n\) be a closed rectangle and let \(f\) and \(g\) be in \(\mathcal{R}(R)\) and \(\alpha \in \mathbb{R}\).

i. \(\alpha f\) is in \(\mathcal{R}(R)\) and \(\int_R \alpha f = \alpha \int_R f\)
ii. \(f+g\) is in \(\mathcal{R}(R)\) and \(\int_R (f+g) = \int_R f + \int_R g\).

Let \(R \subset \mathbb{R}^n\) be a closed rectangle and let \(f\) and \(g\) be in \(\mathcal{R}(R)\) and let \(f(x) \leq g(x)\) for all \(x \in R\). Then \(\int_R f \leq \int_R g\).

Again for simplicity if \(f \colon S \to \mathbb{R}\) is a function and \(R \subset S\) is a closed rectangle, then if the restriction \(f|_R\) is integrable we say \(f\) is integrable on \(R\), or \(f\) in \(\mathcal{R}(R)\) and we write \(\int_R f := \int_R f|_R\).

For a closed rectangle \(S \subset \mathbb{R}^n\), if \(f \colon S \to \mathbb{R}\) is integrable and \(R \subset S\) is a closed rectangle, then \(f\) is integrable over \(R\).

Given \(\epsilon > 0\), we find a partition \(P\) such that \(U(P,f) - L(P,f) < \epsilon\). By making a refinement of \(P\) we can assume that the endpoints of \(P\) are in \(P\), or in other words, \(R\) is a union of subrectangles of \(P\). Then the subrectangles of \(P\) divide into two collections, ones that are subsets of \(R\) and ones whose intersection with the interior of \(R\) is empty. Suppose that \(R_1, R_2, \ldots, R_K\) be the subrectangles that are subsets of \(R\) and \(R_1, R_2, \ldots, R_K\) be the rest. Let \(\tilde{P}\) be the partition of \(R\) composed of those subrectangles of \(P\) contained in \(R\). Then using the same notation as before. \(\sum_{k=1}^K (M_k-m_k) V(R_k) + \sum_{k=K+1}^N (M_k-m_k) V(R_k) \leq \epsilon\) \& \(\geq \text{sum}_{k=1}^K (M_k-m_k) V(R_k) = U(\tilde{P}, f) - L(\tilde{P}, f) \leq \epsilon\). Therefore \(f\) is integrable.

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**Integrals of continuous functions**

FIXME: We will later on prove a much more general result, but it is useful to start with continuous functions only. Before we
get to continuous functions, let us state the following proposition, which has a very easy proof, but it is useful to emphasize as a technique.

Let \( R \subset \mathbb{R}^n \) be a closed rectangle and \( f \colon R \to \mathbb{R} \) a bounded function. If for every \( \epsilon > 0 \), there exists a partition \( \{P\} \) of \( R \) such that \( |U(P,f) - L(P,f)| < \epsilon \) then \( f \in \mathcal{R}(R) \).

Given an \( \epsilon > 0 \) find \( \{P\} \) as in the hypothesis. Then \( |\overline{\int_R} f - \underline{\int_R} f| \leq U(P,f) - L(P,f) < \epsilon \) As \( |\overline{\int_R} f - \underline{\int_R} f| \geq |\overline{\int_R} f - \underline{\int_R} f| \) and the above holds for every \( \epsilon > 0 \), we conclude \( |\overline{\int_R} f - \underline{\int_R} f| \) and \( f \in \mathcal{R}(R) \).

We say a rectangle \( R = [a^1,b^1] \times [a^2,b^2] \times [a^n,b^n] \) has longest side at most \( \alpha \) if \( (b^k-a^k) \leq \alpha \) for all \( k \).

If a rectangle \( R \subset \mathbb{R}^n \) has longest side at most \( \alpha \), then for any \( \epsilon > 0 \), \( \left| \int_R f \right| \leq \epsilon \).

\[
\begin{split}
\left| \int_R f \right| &= \sqrt{\left| \int_R f \right|^2} \\
&\leq \sqrt{\left( \int_R f \right)^2} \\
&\leq \sqrt{\alpha^2} \\
&\leq \sqrt{n} \alpha.
\end{split}
\]

Integration of functions with compact support

Let \( U \subset \mathbb{R}^n \) be an open set and \( f \colon U \to \mathbb{R} \) be a function. We say the support of \( f \) is the set \( \{x \in U : f(x) \neq 0 \} \). That is, the support is the closure of the set of points where the function is nonzero. So for a point not in the support we have that \( f \) is constantly zero in a whole
neighbourhood.

A function \( f \) is said to have \textit{compact support} if \( \text{supp}(f) \) is a compact set. We will mostly consider the case when \( U = \mathbb{R}^n \). In light of the following exercise, this is not an oversimplification.

Suppose \( U \subseteq \mathbb{R}^n \) is open and \( f : U \to \mathbb{R} \) is continuous and of compact support. Show that the function \( \tilde{f} : \mathbb{R}^n \to \mathbb{R} \) defined by:
\[
\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}
\]
is continuous.

Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) be a function with compact support. If \( R \) is a closed rectangle such that \( \text{supp}(f) \subseteq R^o \) where \( R^o \) is the interior of \( R \), and \( f \) is integrable over \( R \), then for any other closed rectangle \( S \) with \( \text{supp}(f) \subseteq S^o \), the function \( f \) is integrable over \( S \) and \( \int_S f = \int_{\tilde{R}} f \).

The intersection of closed rectangles is again a closed rectangle (or empty). Therefore we can take \( \tilde{R} = R \cap S \) to be the intersection of all rectangles containing \( \text{supp}(f) \). If \( \tilde{R} \) is the empty set, then \( \text{supp}(f) \) is the empty set and \( \int f \) is identically zero and the proposition is trivial. So suppose that \( \tilde{R} \) is nonempty. As \( \tilde{R} \subseteq R \) we know that \( f \) is integrable over \( \tilde{R} \). Furthermore \( \tilde{R} \subseteq S \). Given \( \epsilon > 0 \), take \( \tilde{P} \) to be a partition of \( \tilde{R} \) such that
\[
|U(\tilde{P},f_{\tilde{R}}) - L(\tilde{P},f_{\tilde{R}})| < \epsilon.
\]
Now add the endpoints of \( S \) to \( \tilde{P} \) to create a new partition \( P \). Note that the subrectangles of \( \tilde{R} \) are subrectangles of \( P \) as well. Let \( \tilde{R} \cap (R -_{1,2} \text{ldots, } R_K) \) be the subrectangles of \( \tilde{R} \) and \( (R_{K+1}, \text{ldots, } R_N) \) the new subrectangles. Note that since
\[
\text{supp}(f) \subseteq \tilde{R} \text{ and } \tilde{R} \subseteq R_i \subseteq S \text{ for } i = K+1, \text{ldots, } N \text{ we have } \int_{\tilde{R}} f = \int_{(R_{K+1}, \text{ldots, } R_N)} f.
\]
We get \( \int_S f = \int_{(R_{K+1}, \text{ldots, } R_N)} f \), or in other words \( \int_{\tilde{R}} f = \int_{\tilde{R}} f \).

Because of this proposition, when \( f : \mathbb{R}^n \to \mathbb{R}^n \) has compact support and is integrable over a rectangle \( R \) containing the support we write \( \int_{\tilde{R}} f = \int_R f \).

\[\text{Exercises}\]

**FIXME**

**FIXME:** Show that integration over a rectangle with one side of size zero results in zero integral.

Suppose \( R' \) and \( R'' \) are two closed rectangles with \( R' \subseteq R'' \). Suppose that \( f : \mathbb{R}^n \to \mathbb{R}^n \) is in \( \mathcal{L}(R') \). Show that \( \int_R f = \int_{R''} f \).

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Suppose \( R \) and \( R' \) are two closed rectangles with \( R' \subset R \). Suppose that \( f \colon R \to \{\mathbb{R}\} \) is in \( \{\text{mathcal} \{R\} \}\(R')\) and \( f(x) = 0 \) for \( x \notin R' \). Show that \( f \in \{\text{mathcal} \{R\}\}\{R\}) \) and \( \int_{R'} f = \int_{R} f \). Hint: see the previous exercise.

Prove a stronger version of . Suppose \( f \colon \{\text{mathbb} \{R\}\}^n \to \{\text{mathbb} \{R\}\} \) be a function with compact support. Prove that if \( R \) is a closed rectangle such that \( \{\text{operatorname} \{\text{supp}\}(f) \subset R \) and \( f \) is integrable over \( R \), then for any other closed rectangle \( S \) with \( \{\text{operatorname} \{\text{supp}\}(f) \subset S \), the function \( f \) is integrable over \( S \) and \( \int_{S} f = \int_{R} f \). Hint: notice that now the new rectangles that you add as in the proof can intersect \( \{\text{operatorname} \{\text{supp}\}(f) \) on their boundary.

Suppose that \( R \) and \( S \) are closed rectangles. Let \( f(x) := 1 \) if \( x \in R \) and \( f(x) = 0 \) otherwise. Show that \( f \) is integrable over \( S \) and compute \( \int_{S} f \).