6.2: Orbits and Stabilizers

In this section, we'll examine orbits and stabilizers, which will allow us to relate group actions to our previous study of cosets and quotients.

Definition 6.1.0: The Orbit

Let \( S \) be a \( G \)-set, and \( s \in S \). The orbit of \( s \) is the set \( G \cdot s = \{ g \cdot s \mid g \in G \} \), the full set of objects that \( s \) is sent to under the action of \( G \).

There are a few questions that come up when encountering a new group action. The foremost is 'Given two elements \( s \) and \( t \) from the set \( S \), is there a group element such that \( g \cdot s = t \)?' In other words, can I use the group to get from any element of the set to any other? In the case of the action of \( S_n \) on a coin, the answer is yes. But in the case of \( S_4 \) acting on the deck of cards, the answer is no. In fact, this is just a question about orbits. If there is only one orbit, then I can always find a group element to move from any object to any other object. This case has a special name.

Definition 6.1.1: Transitive Group Action

A group action is transitive if \( G \cdot s = S \). In other words, for any \( s, t \in S \), there exists \( g \in G \) such that \( g \cdot s = t \). Equivalently, \( S \) contains a single orbit.

Equally important is the stabilizer of an element, the subset of \( G \) which leaves a given element \( s \) alone.

Definition 6.1.2: The Stabilizer

The stabilizer of \( s \) is the set \( G_s = \{ g \in G \mid g \cdot s = s \} \), the set of elements of \( G \) which leave \( s \) unchanged under the action.
For example, the stabilizer of the coin with heads (or tails) up is \(\langle A_n \rangle\), the set of permutations with positive sign. In our example with \(\langle S_4 \rangle\) acting on the small deck of eight cards, consider the card \(\langle 4D \rangle\). The stabilizer of \(\langle 4D \rangle\) is the set of permutations \(\langle \sigma \rangle\) with \(\langle \sigma(4) = 4 \rangle\); there are six such permutations.

In both of these examples, the stabilizer was a subgroup; this is a general fact!

**Proposition 6.1.3**

The stabilizer \(\langle G_s \rangle\) of any element \(\langle s \in S \rangle\) is a subgroup of \(\langle G \rangle\).

**Proof 6.1.4**

Let \(\langle g, h \in G_s \rangle\). Then \(\langle gh \cdot s = g \cdot (h \cdot s) = g \cdot s = s \rangle\). Thus, \(\langle gh \in G_s \rangle\). If \(\langle g \in G_s \rangle\), then so is \(\langle g^{-1} \rangle\): By definition of a group action, \(\langle 1 \in G_s \rangle\), so:
\[
\langle s = 1 \cdot s = g^{-1} \cdot s \rangle.
\]
Thus, \(\langle G_s \rangle\) is a subgroup.

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**Group action morphisms**

And now some algebraic examples!

1. Let \(\langle G \rangle\) be any group and \(\langle S = G \rangle\). The **left regular action** of \(\langle G \rangle\) on itself is given by left multiplication: \(\langle g \cdot h = gh \rangle\). The first condition for a group action holds by associativity of the group, and the second condition follows from the definition of the identity element. (There is also a **right regular action**, where \(\langle g \cdot h = hg \rangle\); the action is ‘on the right’. The Cayley graph of the left regular action is the same as the usual Cayley graph of the group!

2. Let \(\langle H \rangle\) be a subgroup of \(\langle G \rangle\), and let \(\langle S \rangle\) be the set of cosets \(\langle G \backslash \text{mathord H} \rangle\). The **coset action** is given by \(\langle g \cdot (xH) = (gx)H \rangle\).

![Figure 6.1: \(\langle H \rangle\) is the subgroup of \(\langle S_4 \rangle\) with \(\langle \text{sigma}(1) = 1 \rangle\) for all \(\langle \text{sigma} \rangle\) in \(\langle H \rangle\). This illustrates the action of \(\langle S_4 \rangle\) on cosets of \(\langle H \rangle\).](image)

**Exercise 6.1.5**
Consider the permutation group \( S_n \), and fix a number \( i \) such that \( 1 \leq i \leq n \). Let \( H_i \) be the set of permutations in \( S_n \) with \( \sigma(i) = i \).

1. Show \( H_i \) is a subgroup of \( S_n \).
2. Now let \( n = 5 \) and Sketch the Cayley graph of the coset action of \( S_5 \) on \( H_1 \) and \( H_3 \).

The coset action is quite special; we can use it to get a general idea of how group actions are put together.

**Proposition 6.1.6**

Let \( S \) be a \( G \)-set, with \( s \in S \) and \( G_s \). For any \( g, h \in G \), \( g \cdot s = h \cdot s \) if and only if \( gG_s = hG_s \).

As a result, there is a bijection between elements of the orbit of \( s \) and cosets of the stabilizer \( G_s \).

**Proof 6.1.7**

We have \( gG_s = hG_s \) if and only if \( h^{-1}g \in G_s \), if and only if \( (h^{-1}g) \cdot s = s \), if and only if \( h \cdot s = g \cdot s \), as desired.

In fact, we can generalize this idea considerably. We're actually identifying elements of the \( G \)-set with cosets of the stabilizer group, which is also a \( G \)-set; in other words, defining a function \( \phi \) between two \( G \)-sets. The theorem says that this function preserves the group operation: \( \phi(g \cdot s) = g \cdot \phi(s) \).

**Definition**

Let \( \langle S, T \rangle \) be \( \langle G \rangle \)-sets. A **morphism of \( G \)-sets** is a function \( \phi : S \rightarrow T \) such that \( \phi(g \cdot s) = g \cdot \phi(s) \) for all \( g \in G, s \in S \). We say the \( G \)-sets are **isomorphic** if \( \phi \) is a bijection.

We can then restate the proposition:

**Theorem 6.1.9**

For any \( s \) in a \( G \)-set \( S \), the orbit of \( s \) is isomorphic to the coset action on \( G_s \).

Now we can use LaGrange's theorem in a very interesting way! We know that the cardinality of a subgroup divides the order of the group, and that the number of cosets of a subgroup \( H \) is equal to \( |G|/|H| \). Then we can use the relationship between cosets and orbits to observe the following:

**Theorem 6.1.10**
Let \( S \) be a \( (G) \)-set, with \( s \in S \). Then the size of the orbit of \( s \) is \(|G|/|G_s|\).

For a somewhat obvious example, considering \( (S_{13}) \) acting on the numerical values of playing cards, we can observe that any given card is fixed by a subgroup of \( (S_{13}) \) isomorphic to \( (S_{12}) \) (switching around the other twelve numbers in any way doesn't change affect the given card). Then the size of the orbit of the card is \(|(S_{13})/|G_s|=13\). That's a number we could have figured out directly by reasoning a bit, but it shows us that the theorem is working sensibly!

Now that we have a notion of isomorphism of \( (G) \)-sets, we can say something to classify \( (G) \)-sets. What kinds of actions are possible?

Let \( (G) \) be a finite group, and \( (S) \) a finite \( (G) \)-set. Then \( (S) \) is a collection of orbits. We know that every orbit is isomorphic to \( (G) \) acting on the cosets of some subgroup of \( (H) \). So we have the following theorem:

**Theorem 6.1.11: Classification of \( (G) \)-Sets**

Let \( (G) \) be a finite group, and \( (S) \) a finite \( (G) \)-set. Then \( (S) \) is isomorphic to a union of coset actions of \( (G) \) on subgroups.

For example, \( (S_{13}) \) acting on a full deck of cards decomposes as a union of four orbits, each isomorphic to the coset action of \( (S_{13}) \) on a subgroup isomorphic to \( (S_{12}) \).

In short, to understand all possible \( (G) \)-sets, we should try to understand all of the subgroups of \( (G) \). In general, this is a hard problem, though it's easy for some cases.

**Exercise 6.1.12**

1. For \( n=15 \), draw Cayley graphs of the coset action of \( (\mathbb{Z}_{15}) \) on each of it's cosets.
2. Describe all the subgroups of \( (\mathbb{Z}_{n}) \) for arbitrary \( n \).

**Exercise 6.1.13**

\( (S_{n}) \) acts on subsets of \( (N=\{1,2,3,\ldots,n\}) \) in a natural way: if \( \{i_1, \ldots, i_k\} \subseteq N \), then \( \{\sigma(i_1), \ldots, \sigma(i_k)\} \).

1. Decompose the action of \( (S_{4}) \) on the subsets of \( (\{1,2,3,4\}) \) into orbits.
2. Draw a Cayley graph of the action.
3. Identify each orbit with the coset action on a subgroup of \( (S_{4}) \).

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