5.3: DeMoivre’s Theorem and Powers of Complex Numbers

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is de Moivre’s Theorem and why is it useful?
- If \(n\) is a positive integer, what is an \(n\)th root of a complex number? How many \(n\)th roots does a complex number have? How do we find all of the \(n\)th roots of a complex number?

The trigonometric form of a complex number provides a relatively quick and easy way to compute products of complex numbers. As a consequence, we will be able to quickly calculate powers of complex numbers, and even roots of complex numbers.

Beginning Activity

Let \(z = r(\cos(\theta) + i\sin(\theta))\). Use the trigonometric form of \(z\) to show that

\[
[z^2] = r^2(\cos(2\theta) + i\sin(2\theta)) \tag{eq1}
\]

De Moivre’s Theorem

The result of Equation \ref{eq1} is not restricted to only squares of a complex number. If \(z = r(\cos(\theta) + i\sin(\theta))\), then it is also true that
\[
\begin{align*}
z^3 &= zz^2 \\
&= (r)(r^2)(\cos(\theta + 2\theta) + i\sin(\theta + 2\theta)) \\
&= r^3(\cos(3\theta) + i\sin(3\theta))
\end{align*}
\]

We can continue this pattern to see that
\[
\begin{align*}
z^4 &= zz^3 \\
&= (r)(r^3)(\cos(\theta + 3\theta) + i\sin(\theta + 3\theta)) \\
&= r^4(\cos(4\theta) + i\sin(4\theta))
\end{align*}
\]

The equations for \((z^2)^2\), \((z^3)^3\), and \((z^4)^4\) establish a pattern that is true in general; this result is called de Moivre’s Theorem.

DeMoivre’s Theorem

Let \((z = r(\cos(\theta) + i\sin(\theta)))\) be a complex number and \((n)\) any integer. Then
\[
[z^n] = (r^n)(\cos(n\theta) + i\sin(n\theta)) \label{DeMoivre}
\]

It turns out that DeMoivre’s Theorem also works for negative integer powers as well.

Exercise \((PageIndex{1})\)

Write the complex number \((1 - i)\) in polar form. Then use DeMoivre’s Theorem (Equation \ref{DeMoivre}) to write \(((1 - i)^10)\) in the complex form \((a + bi)\), where \((a)\) and \((b)\) are real numbers and do not involve the use of a trigonometric function.

\textbf{Answer}

In polar form,
\[
[1 - i = \sqrt{2}(\cos(-\dfrac{\pi}{4}) + \sin(-\dfrac{\pi}{4}))]
\]

So \(((1 - i)^10) = (\sqrt{2})^10(\cos(-\dfrac{10\pi}{4}) + \sin(-\dfrac{10\pi}{4})) = 32(\cos(-\dfrac{5\pi}{2}) + \sin(-\dfrac{5\pi}{2})) = 32(0 - i) = -32i\]

\textbf{Roots of Complex Numbers}

DeMoivre’s Theorem is very useful in calculating powers of complex numbers, even fractional powers. We illustrate with an example.

Example \((PageIndex{1})\): Roots of Complex Numbers

We will find all of the solutions to the equation \((x^3 - 1 = 0)\). These solutions are also called the \textit{roots} of the polynomial \((x^3 - 1)\).
Solution

To solve the equation \(x^3 - 1 = 0\), we add 1 to both sides to rewrite the equation in the form \(x^3 = 1\). Recall that to solve a polynomial equation like \(x^3 = 1\) means to find all of the numbers (real or complex) that satisfy the equation. We can take the real cube root of both sides of this equation to obtain the solution \(x_0 = 1\), but every cubic polynomial should have three solutions. How can we find the other two? If we draw the graph of \(y = x^3 - 1\) we see that the graph intersects the \((x,\) axis at only one point, so there is only one real solution to \(x^3 = 1\). That means the other two solutions must be complex and we can use DeMoivre’s Theorem to find them. To do this, suppose

\[
[z = r[\cos(\theta) + i\sin(\theta)]]\text{ is a solution to } (x^3 = 1)\text{. Then}
\]

\[
[1 = z^3 = r^3[\cos(3\theta) + i\sin(3\theta)]]. \text{ nonumber }
\]

This implies that \(r = 1\) (or \(r = -1\)), but we can incorporate the latter case into our choice of angle). We then reduce the equation \((x^3 = 1)\) to the equation

\[
[1 = \cos(3\theta) + i\sin(3\theta)]
\]

has solutions when \(\cos(3\theta) = 1\) and \(\sin(3\theta) = 0\). This will occur when \(3\theta = 2\pi k\), or \(\theta = \frac{2\pi k}{3}\), where \(k\) is any integer. The distinct integer multiples of \(\frac{2\pi}{3}\) on the unit circle occur when \(k = 0\) and \(\theta = 0\), \(k = 1\) and \(\theta = \frac{2\pi}{3}\), and \(k = 2\) with \(\theta = \frac{4\pi}{3}\). In other words, the solutions to \((x^3 = 1)\) should be

\[
\begin{align*}
x_{0} &= \cos(0) + i\sin(0) = 1 \\
x_{1} &= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\
x_{2} &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i
\end{align*}
\]

We already know that \((x^3 = 1)\) so \((x_{0})\) actually is a solution to \((x^3 = 1)\). To check that \((x_{1})\) and \((x_{2})\) are also solutions to \((x^3 = 1)\), we apply DeMoivre’s Theorem (Equation \ref{DeMoivre}):

\[
\begin{align*}
(x^3 = 1)_{1} &= [\cos(2\pi) + i\sin(2\pi)]^3 = \cos(3\cdot2\pi) + i\sin(3\cdot2\pi) = 1 \\
(x^3 = 1)_{2} &= [\cos(4\pi) + i\sin(4\pi)]^3 = \cos(3\cdot4\pi) + i\sin(3\cdot4\pi) = 1
\end{align*}
\]

Thus, \((x^3 = 1)_{1}\) and \((x^3 = 1)_{2}\) and we have found three solutions to the equation \((x^3 = 1)\). Since a cubic can have only three solutions, we have found them all.

The general process of solving an equation of the form \((x^n = a + bi)\), where \(n\) is a positive integer and \((a + bi)\) is a complex number works the same way. Write \((a + bi)\) in trigonometric form

\[
[a + bi = r[\cos(\theta) + i\sin(\theta)]]. \text{ nonumber }
\]

and suppose that \((z = s[\cos(\alpha) + i\sin(\alpha)])\) is a solution to \((x^n = a + bi)\). Then
Using the last equation, we see that
\[ |s^n| = |r| \] and \[ \cos(\theta) + i\sin(\theta) = \cos(n\alpha) + i\sin(n\alpha) \]

Therefore, \[ |s^n| = |r| \] and \[ n\alpha = \theta + 2\pi k \]

where \( k \) is any integer. This give us
\[ s = \sqrt[n]{r} \] and \[ \alpha = \dfrac{\theta + 2\pi k}{n} \]

We will get n different solutions for \( k = 0, 1, 2, ..., n - 1 \), and these will be all of the solutions. These solutions are called the \( n \)th roots of the complex number \( a + bi \). We summarize the results.

If we want to represent the \( n \)th roots of \( r[\cos(\theta) + i\sin(\theta)] \) using degrees instead of radians, the roots will have the form
\[ \sqrt[n]{r} \left[ \cos\left(\dfrac{\theta + 360^\circ k}{n}\right) + i\sin\left(\dfrac{\theta + 360^\circ k}{n}\right) \right] \]

for \( k = 0, 1, 2, ..., (n - 1) \).

Roots of Complex Numbers

Let \( n \) be a positive integer. The \( n \)th roots of the complex number \( r[\cos(\theta) + i\sin(\theta)] \) are given by
\[ \sqrt[n]{r} \left[ \cos\left(\dfrac{\theta + 2\pi k}{n}\right) + i\sin\left(\dfrac{\theta + 2\pi k}{n}\right) \right] \]

for \( k = 0, 1, 2, ..., (n - 1) \).

Example \( \PageIndex{2} \): Square Roots of 1

As another example, we find the complex square roots of 1. In other words, we find the solutions to the equation \( z^2 = 1 \).

Of course, we already know that the square roots of \( 1 \) are \( 1 \) and \( -1 \), but it will be instructive to utilize our general result and see that it gives the same result. Note that the trigonometric form of \( 1 \) is
\[ 1 = \cos(0) + i\sin(0) \]

so the two square roots of \( 1 \) are
\[ \sqrt{1} \left[ \cos\left(\dfrac{0 + 2\pi(0)}{2}\right) + i\sin\left(\dfrac{0 + 2\pi(0)}{2}\right) \right] = \cos(0) + i\sin(0) = 1 \]
\[ \sqrt{1} \left[ \cos(\frac{0 + 2\pi(1)}{2}) + i\sin(\frac{0 + 2\pi(1)}{2}) \right] = \cos(\pi) + i\sin(\pi) = -1 \]

as expected.

Exercise \( \PageIndex{2} \)

1. Find all solutions to \( x^4 = 1 \). (The solutions to \( x^n = 1 \) are called the \( n \)th roots of unity, with unity being the number 1.)
2. Find all sixth roots of unity.

Answer

1. We find the solutions to the equation \( z^4 = 1 \). Let \( \omega = \cos(\frac{2\pi}{4}) + i\sin(\frac{2\pi}{4}) = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) \). Then
   - \( \omega^0 = 1 \),
   - \( \omega = i \),
   - \( \omega^2 = \cos(\frac{2\pi}{2}) + i\sin(\frac{2\pi}{2}) = -1 \)
   - \( \omega^3 = \cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2}) = -i \)

So the four fourth roots of unity are \( 1, i, -1, -i \).

2. We find the solutions to the equation \( z^6 = 1 \). Let \( \omega = \cos(\frac{2\pi}{6}) + i\sin(\frac{2\pi}{6}) = \cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}) \). Then
   - \( \omega^0 = 1 \),
   - \( \omega = \frac{1}{2} + \sqrt{3}i \),
   - \( \omega^2 = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}) = -\frac{1}{2} + \sqrt{3}i \)
   - \( \omega^3 = \cos(\frac{3\pi}{3}) + i\sin(\frac{3\pi}{3}) = -1 \)
   - \( \omega^4 = \cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3}) = -\frac{1}{2} - \sqrt{3}i \)
   - \( \omega^5 = \cos(\frac{5\pi}{3}) + i\sin(\frac{5\pi}{3}) = \frac{1}{2} - \sqrt{3}i \)

So the four fourth roots of unity are \( 1, \frac{1}{2} + \sqrt{3}i, -\frac{1}{2} + \sqrt{3}i, -1, -\frac{1}{2} - \sqrt{3}i, \frac{1}{2} - \sqrt{3}i \).

Now let’s apply our result to find roots of complex numbers other than \( 1 \).

Example \( \PageIndex{3} \): Roots of Other Complex Numbers

We will find the solutions to the equation

\[ x^4 = -8 + 8\sqrt{3}i \]

Solution

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Note that we can write the right hand side of this equation in trigonometric form as
\[-8 + 8\sqrt{3}i = 16(\cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})).\]

The fourth roots of \(-8 + 8\sqrt{3}i\) are then
\[
\begin{align*}
x_0 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3} + 2\pi(0)}{4}\right) + i\sin\left(\frac{\frac{2\pi}{3} + 2\pi(0)}{4}\right)\right] = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right] = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i, \\
x_1 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3} + 2\pi(1)}{4}\right) + i\sin\left(\frac{\frac{2\pi}{3} + 2\pi(1)}{4}\right)\right] = 2\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + \sqrt{3}i, \\
x_2 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3} + 2\pi(2)}{4}\right) + i\sin\left(\frac{\frac{2\pi}{3} + 2\pi(2)}{4}\right)\right] = 2\left[\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right] = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i, \\
x_3 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3} + 2\pi(3)}{4}\right) + i\sin\left(\frac{\frac{2\pi}{3} + 2\pi(3)}{4}\right)\right] = 2\left[\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right] = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}i.
\end{align*}
\]

Exercise \(\PageIndex{3}\)

Find all fourth roots of \(-256\), that is find all solutions of the equation \(x^4 = -256\).

\textbf{Answer}

Since \((-256 = 256\cos(\pi) + i\sin(\pi))\) we see that the fourth roots of \((-256)\) are
\[
\begin{align*}
x_0 &= \sqrt[4]{256}\left[\cos\left(\frac{\pi + 2\pi(0)}{4}\right) + i\sin\left(\frac{\pi + 2\pi(0)}{4}\right)\right] = 4\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = 4\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = 2\sqrt{2} + 2i\sqrt{2}, \\
x_1 &= \sqrt[4]{256}\left[\cos\left(\frac{\pi + 2\pi(1)}{4}\right) + i\sin\left(\frac{\pi + 2\pi(1)}{4}\right)\right] = 4\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = 4\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = -2\sqrt{2} + 2i\sqrt{2}, \\
x_2 &= \sqrt[4]{256}\left[\cos\left(\frac{\pi + 2\pi(2)}{4}\right) + i\sin\left(\frac{\pi + 2\pi(2)}{4}\right)\right] = 4\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = 4\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -2\sqrt{2} - 2i\sqrt{2}, \\
x_3 &= \sqrt[4]{256}\left[\cos\left(\frac{\pi + 2\pi(3)}{4}\right) + i\sin\left(\frac{\pi + 2\pi(3)}{4}\right)\right] = 4\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) = 4\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = 2\sqrt{2} - 2i\sqrt{2}.
\end{align*}
\]

\textbf{Summary}

\textbf{In this section, we studied the following important concepts and ideas:}

\textbf{DeMoivre's Theorem}

Let \(z = r(\cos(\theta) + i\sin(\theta))\) be a complex number and \(n\) any integer. Then
\[|z^n| = (r^n)(\cos(n\theta) + i\sin(n\theta)) \text{ \ \nonumber} \]
Roots of Complex Numbers

Let \( n \) be a positive integer. The \( n \)th roots of the complex number \( r[\cos(\theta) + i\sin(\theta)] \) are given by

\[
\sqrt[n]{r} \left[ \cos \left( \frac{\theta + 2\pi k}{n} \right) + i\sin \left( \frac{\theta + 2\pi k}{n} \right) \right]
\]

for \( k = 0, 1, 2, ..., (n - 1) \).