5.1: Linear Diophantine Equations

Thinking out loud

Mary went to a park and saw vehicles with \(2\) wheels and \(4\) wheels. She counted the wheels. When she came home she told her mom that the vehicles she had seen had a total of \(28\) wheels. Her mom asked how many vehicles had \(2\) wheels and how many vehicles had \(4\) wheels. What was Mary's response?

Diophantine Equation

A Diophantine equation is a polynomial equation with 2 or more integer unknowns.

A Linear Diophantine equation (LDE) is an equation with 2 or more integer unknowns and the integer unknowns are each to at most degree of 1.

Linear Diophantine equation in two variables takes the form of \(ax+by=c\), where \(x, y \in \mathbb{Z}\) and \(a, b, c\) are integer constants. \(x\) and \(y\) are unknown variables.

A **Homogeneous** Linear Diophantine equation (HLDE) is \(ax+by=0\), \(x, y \in \mathbb{Z}\). Note that \(x=0\) and \(y=0\) is a solution, called the trivial solution for this equation.

Example \(\PageIndex{1}\):

Example of a homogeneous linear diophantine equation:

\(5x-3y=0, x, y \in \mathbb{Z}\).
In this case \( (x=3), (y=5) \) is a solution as is \( (x=6), (y=10) \).
Hence \((x=3k), (y=5k), k \in \mathbb{Z}\) represent all the solutions.
Check: \( 5(3k)-3(5k)=15k-15k=0. \)

**** NOTE **** In a homogeneous linear diophantine equation, the minute the equation is an addition, one of the variable is required to be negative.
In the case of \((5x+3y=0, x, y \in \mathbb{Z})\), \((x=-3k), (y=5k), k \in \mathbb{Z}\) are solutions.

THEOREM: Homogeneous Linear Diophantine Equation

Let \( ax+by=0, x, y \in \mathbb{Z} \) be a homogeneous linear Diophantine equation.
If \( \gcd(a, b)=d \), then the complete family of solutions to the above equation is
\( x=\frac{b}{d} k, y=-\frac{a}{d} k, k \in \mathbb{Z} \).

Example \( \PageIndex{2} \): Solve the Homogeneous linear Diophantine equation
\( 6x+9y=0, x, y \in \mathbb{Z} \).

Solution:
Note that GCD of 6 and 9 is 3. Hence the solutions are
\( x=\frac{9k}{3}=3k \) and \( y=-\frac{6k}{3}=-2k \) with \( k \in \mathbb{Z} \).

Use the following steps to solve a non-homogeneous linear Diophantine equation.

Solve the linear Diophantine Equations: \( ax+by=c, x, y \in \mathbb{Z} \).

Use the following steps to solve a non-homogeneous linear Diophantine equation.

Step 1: Determine the GCD of a and b. Let suppose \( \gcd(a, b)=d \).
Step 2: Check that the GCD of a and b divides c. NOTE: If YES, continue on to step 3. If NO, STOP as there are no solutions.
Step 3: Find a particular solution to \( ax+by=c \) by first finding \( (x_0, y_0) \) such that \( ax+by=d \). Suppose
\( x=\frac{c}{d} x_0 \) and \( y=\frac{c}{d} y_0 \).
Step 4: Use a change of variables: Let \( u=x-\frac{c}{d} x_0 \) and \( v=y-\frac{c}{d} y_0 \), then we will see that \( au+bv=0 \) (important to check your result).
Step 5: Solve \( au+bv=0 \). That is: \( u=\frac{c}{d} b \) and \( v=\frac{c}{d} a \) with \( m \in \mathbb{Z} \).
Step 6: Substitute for \( u \) and \( v \). Thus the general solutions are
\( x=\frac{c}{d} x_0, y=\frac{c}{d} y_0 \) and \( x=\frac{c}{d} x_0=n, y=\frac{c}{d} y_0=m \) with \( n, m \in \mathbb{Z} \).

Example \( \PageIndex{3} \): Die hard Jug Problem

Solve the linear Diophantine Equations: \( 5x+3y=4, x, y \in \mathbb{Z} \).

Solution:
Step 1: Determine the GCD of 5 and 3 (a and b). Since \( \gcd(5, 3)=1 \), \( \gcd(5, 3)=1 \).
Step 2: Since \(1 \mid 4\), we will continue on to Step 3.

Step 3: Find a particular solution to \((5x+3y=4, x,y \in \mathbb{Z})\).

Since \((5(5)+3(-7)=4, x=5)\) and \((y=-7)\) is a particular solution.

Step 4: Let \((u=x-5)\) and \((v=y+7)\) Note: The opposite integer of Step 4, so if it's positive in step 4 it will be negative in step 5 and vice versa.

Then \((5u+3v=5(x-5)+3(y+7))\)
\((-5x+25+3y+21)\)
\((-5x+3y+4)\) (because the equation is \((5x+3y=4)\))
\((-0)\)

Step 5: Solve \(5u+3v=0\)

The general solutions are \((u=-3m)\) and \((v=5m, m \in \mathbb{Z})\).

Step 6: \((x-5=-3m)\) and \((y+7=5m, m \in \mathbb{Z})\).

Hence the general solutions are \((x=-3m+5, y=5m-7, m \in \mathbb{Z})\).

Example \(\PageIndex{4}\):

Solve the linear Diophantine Equations: \((2x+4y=21, x,y \in \mathbb{Z})\).

Solution:
Since \((\gcd(2, 4)=2)\) and \((2)\) does not divide \((21)\), \((2x+4y=21)\) has no solution.

Example \(\PageIndex{5}\)

Solve the linear Diophantine Equation \((20x+16y=500, x,y \in \mathbb{Z}_+))\).

Solution

Both \((x, y \geq 0, 500 = 20(x) + 16(y))\)

Step 1: \((\gcd(20, 16) = 4)\) Since \((4 \mid 500)\), we expect a solution.

Step 2: A solution is \((4125=20(1)(125)+16(-1)(125))\)

\((500=20(125)+16(-125))\)

Hence, \((x = 125)\) and \((y = -125)\) is a solution to \((500 = 20x + 16y)\).
Step 3: Let \( u = x - 125 \) and \( v = y + 125 \).

Consider that 
\[
20u + 16v = 20x - (20)(125) + 16y + (16)(125)
\]
\[
= 20x + 16y - [(20)(125) - (16)(125)]
\]
\[
= 20x + 16y - 500.
\]

Thus, \( 20u + 16v = 0 \).

Step 4: In general, the solution to \( ax + by = 0 \) is \( x = bk \), \( y = -ak \), \( k \in \mathbb{Z} \setminus \{0\} \), \( d = \gcd(a,b) \). Recall, \( \gcd(20, 16) = 4 \).

Thus \( u = 16k/4 = 4k \) and \( v = -20k/4 = -5k \), \( k \in \mathbb{Z} \).

Step 5: Replace \( u \) and \( v \).

Consider \( 4k = x - 125 \) and \( -5k = y + 125 \).

Hence, \( x = 4k + 125 \) and \( y = -5k - 125 \).

Step 6: Both \( x \) and \( y \geq 0 \). \( x \leq 25 \) and \( y \leq 31 \) since total is 500.

\[
4k + 125 \geq 0, \quad k \geq -125/4, \quad k \geq -31.25.
\]
\[
4k + 125 \leq 25, \quad 4k \leq -100, \quad k \leq -25.
\]

Thus, the possible solutions are:

Let \( k = -25 \) then \( x = 25, y = 0 \).

Let \( k = -26 \) then \( x = 21, y = 5 \).

Let \( k = -27 \) then \( x = 17, y = 10 \).

Let \( k = -28 \) then \( x = 13, y = 15 \).

Let \( k = -29 \) then \( x = 9, y = 20 \).

Let \( k = -30 \) then \( x = 5, y = 25 \).

Let \( k = -31 \) then \( x = 1, y = 30 \).

Thus the options of \( \{(x,y)\} \) that satisfy the given equation are:

\[
\{(25,0), (21,5), (17,10), (13, 15), (9, 20), (5, 25), (1,30)\}
\]
The following problem can be found in puzzle books.

Example \( \PageIndex{6} \)

When Mrs. Brown cashed her cheque, the absent minded teller gave her as many cents as she should have dollars, and as many dollars as she should have cents. Equally absent minded Mrs. Brown left with the cash without noticing the discrepancy. It was only after she spent 5 cents that she noticed now she had twice as much money as she should. What was the amount of her cheque?

**Solution**

Let \( x \) be the number of dollars Mrs Brown should have received and \( y \) be the number of cents she should have received.

Then \( 2(100x + y) = 100y + x - 5 \)

Note double original amount without spending a nickel.

\[ 200x + 2y = 100y + x - 5 \]

\[ 199x - 98y = -5. \]

\[ 5 = 199x + 98y \]

**Step 1:** \( \gcd(199, 98) = 1 \). Since \( 1 \mid 5 \), we can continue.

**Step 2:** A solution is \( 51 = -199(-33)(5) + (98)(-67)(5) \)

\[ 5 = -199(-165) + 98(-335). \]

Hence \( x = -165 \) and \( y = -335 \) is a solution to \( 5 = 98y - 199x \).

**Step 3:** Let \( u = x + 165 \) and \( v = y + 335 \).

Consider that \( -199u + 98v = -199(x + 165) + 98(y + 335) \)

\[ = -199x + 98y - [(199)(165) + (98)(335)] \]

Thus \( -199u + 98v = -199x + 98y - 5 = 0. \)

**Step 4:** In general, the solution to \( ax + by = 0 \) is \( x=bdk \) and \( y=-adk, k \in \mathbb{Z} \setminus \{0\}, d=\gcd(a,b) \).

Recall, \( \gcd(199, 98) = 1. \)

Thus, \( u = 98k \) and \( v = 199k, k \in \mathbb{Z} \).

**Step 5:** Replace \( u \) and \( v \).
\(x + 165 = 98k\) and \(y + 335 = 199k\), \(k \geq Z\).

Hence \(x = -165 + 98k\) and \(y = -335 + 199k\).

**Step 6:** Both \(x\) & \(y \geq 0\) and both \(x\), \(y < 100\)

\[-165 + 98k \geq 0\), so \(k \geq 1.68\]

\[-335 + 199k \geq 0\), so \(k \geq 1.68\]

\[-165 + 98k < 100\), \(98k < 265\), \(k < 2.70\]

\[-335 + 199k < 100\), \(199k < 435\), \(k < 2.18\]

Since, \(1.68 \leq k < 2.18\) and \(k \geq Z\), \(k = 2\).

Thus, \(x = 98(2) - 165 = 31\) and \(y = -335 + 199(2) = 63\).

Thus, the cheque was for $31.63.

To verify, the teller gave Mrs. Brown $63.31, she then spent 5 cents, leaving her with $63.26 which is twice the check amount \((2)(\$31.63)=\$63.26\).?

**PRACTICAL USES**

- Cryptography
- Designing different combinations of a variety of elements.