5.1: Linear Diophantine Equations

Thinking out loud

Mary went to a park and saw vehicles with 2 wheels and 4 wheels. She counted the wheels. When she came home she told her mom that the vehicles she had seen had a total of 28 wheels. Her mom asked how many vehicles had 2 wheels and how many vehicles had 4 wheels. What was Mary's response?

Diophantine Equation

A Diophantine equation is a polynomial equation with 2 or more integer unknowns.

A Linear Diophantine equation (LDE) is an equation with 2 or more integer unknowns and the integer unknowns are each to at most degree of 1.

Linear Diophantine equation in two variables takes the form of \(ax+by=c,\) where \((x, y \in \mathbb{Z})\) and \(a, b, c\) are integer constants. \(x\) and \(y\) are unknown variables.

A **Homogeneous** Linear Diophantine equation (HLDE) is \((ax+by=0, x, y \in \mathbb{Z}).\) Note that \((x=0)\) and \((y=0)\) is a solution, called the trivial solution for this equation.

Example \(
\PageIndex{1}\):

Example of a homogeneous linear diophantine equation:

\((5x-3y=0, x, y \in \mathbb{Z}).\)
In this case \((x= 3), (y=5)\) is a solution as is \((x=6), (y=10)\).
Hence \((x=3k), (y=5k)\) and \((x= -3k), (y= 5k)\) represent all the solutions.
Check: \((5(3k))-3(5k)=15k-15k = 0.\)

**** NOTE**** In a homogeneous linear diophantine equation, the minute the equation is an addition, one of the variable is required to be negative.
In the case of \((5x+3y=0, x, y \in \mathbb{Z})\), \((x= -3k), (y= 5k)\) are solutions.

THEOREM: Homogeneous Linear Diophantine Equation

Let \((ax+by=0, x, y \in \mathbb{Z})\) be a homogeneous linear Diophantine equation.
If \(\gcd(a, b)=d\), then the complete family of solutions to the above equation is \((x=\frac{b}{d} k, y=-\frac{a}{d} k, k \in \mathbb{Z})\).

Example \((\PageIndex{2})\): Solve the Homogeneous linear Diophantine equation
\[(6x+9y=0, x, y \in \mathbb{Z}).\]

Solution:
Note that GCD of 6 and 9 is 3. Hence the solutions are \((x= \frac{9k}{3}=3k)\) and \((y= \frac{-6k}{3}=-2k)\) with \((k \in \mathbb{Z})\).

Use the following steps to solve a non-homogeneous linear Diophantine equation.

Solve the linear Diophantine Equations: \((ax+by=c, x, y \in \mathbb{Z})\).

Use the following steps to solve a non-homogeneous linear Diophantine equation.

Step 1: Determine the GCD of \(a\) and \(b\). Let suppose \((\gcd(a, b)=d)\).
Step 2: Check that the GCD of \(a\) and \(b\) divides \(c\). NOTE: If YES, continue on to step 3. If NO, STOP as there are no solutions.
Step 3: Find a particular solution to \((ax+by=c)\) by first finding \((x_0, y_0)\) such that \((ax+by=d)\). Suppose \((x=\frac{c}{d}x_0)\) and \((y=\frac{c}{d}y_0)\).
Step 4: Use a change of variables: Let \((u=\frac{a}{d}x_0)\) and \((v=\frac{a}{d}y_0)\), then we will see that \((au+bv=0)\) (important to check your result).
Step 5: Solve \((au+bv=0)\). That is: \((u=\frac{v}{a}b\{d\}m)\) and \((v=\frac{v}{a}c\{d\}m, m \in \mathbb{Z}\)).
Step 6: Substitute for \((u)\) and \((v)\). Thus the general solutions are \((x=\frac{c}{d}x_0=\frac{a}{d}b\{d\}m)\) and \((y=\frac{c}{d}y_0=\frac{a}{d}c\{d\}m, m \in \mathbb{Z}\)).

Example \((\PageIndex{3})\): Die hard Jug Problem

Solve the linear Diophantine Equations: \((5x+3y=4, x, y \in \mathbb{Z})\).

Solution:
Step 1: Determine the GCD of 5 and 3 (a and b). Since \((\gcd(5, 3)=1)\), \((\gcd(5, 3)=1).\)
Step 2: Since \(1 \mid 4\), we will continue on to Step 3.

Step 3: Find a particular solution to \(5x+3y=4, x,y \in \mathbb{Z}\).

Since \(5(5)+3(-7)=4, x=5\) and \(y=-7\) is a particular solution.

Step 4: Let \((u=x-5)\) and \((v=y+7)\) Note: The opposite integer of Step 4, so if it's positive in step 4 it will be negative in step 5 and vice versa.

Then \(5u+3v=5(x-5)+3(y+7)\)
\(= 5x-25+3y+21\)
\(=5x+3y-4\)
\(= 4-4\) (because the equation is \(5x+3y=4\))
\(=0.\)

Step 5: Solve \(5u+3v=0\)

The general solutions are \((u=-3m)\) and \((v=5m, m \in \mathbb{Z})\).

Step 6: \((x-5=-3m)\) and \((y+7=5m, m \in \mathbb{Z})\).

Hence the general solutions are \((x=-3m+5, y=5m-7, m \in \mathbb{Z})\).

Example \(\PageIndex{4}\):

Solve the linear Diophantine Equations: \(2x+4y=21, x,y \in \mathbb{Z}\).

Solution:

Since \( \gcd(2, 4)=2\) and \(2\) does not divide \(21\), \(2x+4y=21\) has no solution.

Example \(\PageIndex{5}\)

Solve the linear Diophantine Equation \(20x+16y=500, x,y \in \mathbb{Z}_+\).

Solution

Both \((x, y \geq 0, 500 = 20(x) + 16(y))\)

Step 1: \(( \gcd(20, 16) = 4. \) Since \((4 \mid 500)\), we expect a solution.

Step 2: A solution is \(4125=20(1)(125)+16(-1)(125)\).

\(500=20(125)+16(-125)\)

Hence, \((x = 125)\) and \((y = -125)\) is a solution to \(500 = 20x + 16y\).
Step 3: Let $u = x - 125$ and $v = y + 125$.

Consider that $20u + 16v = 20x - (20)(125) + 16y + (16)(125)$

$= 20x + 16y - [(20)(125) - (16)(125)]$

$= 20x + 16y - 500.$

Thus, $20u + 16v = 0$.

Step 4: In general, the solution to $ax + by = 0$ is $x = bk$ and $y = -dk$, $k \in \mathbb{Z} \setminus \{0\}$, $d = \gcd(a, b)$. Recall, $\gcd(20, 16) = 4$.

Thus $u = 16k/4 = 4k$ and $v = -20k/4 = -5k$, $k \in \mathbb{Z}$.

Step 5: Replace $u$ and $v$.

Consider $4k = x - 125$ and $-5k = y + 125$.

Hence, $x = 4k + 125$ and $y = -5k - 125$.

Step 6: Both $x$ and $y \geq 0$. $x \leq 25$ and $y \leq 31$ since total is 500.

$4k + 125 \geq 0$, $k \geq -125/4$, $k \geq -31.25$.

$4k + 125 \leq 25$, $4k \leq -100$, $k \leq -25$.

Thus, the possible solutions are:

Let $k = -25$ then $x = 25$, $y = 0$.

Let $k = -26$ then $x = 21$, $y = 5$.

Let $k = -27$ then $x = 17$, $y = 10$.

Let $k = -28$ then $x = 13$, $y = 15$.

Let $k = -29$ then $x = 9$, $y = 20$.

Let $k = -30$ then $x = 5$, $y = 25$.

Let $k = -31$ then $x = 1$, $y = 30$.

Thus the options of $\{(x, y)\}$ that satisfy the given equation are:

$\{(25,0), (21,5), (17,10), (13, 15), (9, 20), (5, 25), (1,30)\}$
The following problem can be found in puzzle books.

Example \(\PageIndex{6}\)

When Mrs. Brown cashed her cheque, the absent-minded teller gave her as many cents as she should have dollars, and as many dollars as she should have cents. Equally absent-minded Mrs. Brown left with the cash without noticing the discrepancy. It was only after she spent 5 cents that she noticed now she had twice as much money as she should. What was the amount of her cheque?

**Solution**

Let \(x\) be the number of dollars Mrs Brown should have received and \(y\) be the number of cents she should have received.

Then \(2(100x + y) = 100y + x - 5\)

Note double original amount without spending a nickel.

\[200x + 2y = 100y + x - 5\]

\[199x - 98y = -5\]

\[5 = -199x + 98y\]

**Step 1:** \(\gcd(199,98) = 1\). Since \(1 \mid 5\), we can continue.

**Step 2:** A solution is \(5 = -199(-33)(5) + (98)(-67)(5)\)

\[5 = -199(-165) + 98(-335)\]

Hence \(x = -165\) and \(y = -335\) is a solution to \(5 = 98y - 199x\).

**Step 3:** Let \(u = x + 165\) and \(v = y + 335\).

Consider that \(-199u + 98v = -199(x + 165) + 98(y + 335)\)

\[= -199x + 98y - [(199)(165) + (98)(335)]\]

Thus \(-199u + 98v = -199x + 98y - 5 = 0\).

**Step 4:** In general, the solution to \(ax + by = 0\) is \(x = bdk\) and \(y = -adk\), \(k \in \mathbb{Z}\setminus\{0\}\), \(d = \gcd(a,b)\).

Recall, \(\gcd(199, 98) = 1\).

Thus, \(u = 98k\) and \(v = 199k\), \(k \in \mathbb{Z}\).

**Step 5:** Replace \(u\) and \(v\).
x + 165 = 98k and y + 335 = 199k, k \in \mathbb{Z}.

Hence x = -165 + 98k and y = -335 + 199k.

**Step 6:** Both x & y \geq 0 and both x, y < 100

-165 + 98k \geq 0, so k \geq 1.68

-335 + 199k \geq 0, so k \geq 1.68

-165 + 98k < 100, 98k < 265, ? k < 2.70

-335 + 199k < 100, 199k < 435, ? k < 2.18

Since, 1.68 \leq k < 2.18 and k \in \mathbb{Z}, k = 2.

Thus, x = 98(2) - 165 = 31 and y = -335 + 199(2) = 63.

Thus, the cheque was for $31.63.

To verify, the teller gave Mrs. Brown $63.31, she then spent 5 cents, leaving her with $63.26 which is twice the check amount \((2)(\$31.63)=\$63.26)\).?

**PRACTICAL USES**

- Cryptography
- Designing different combinations of a variety of elements.