1.1: Binary operations

Binary operation

**Definition: Binary operation**

Let \((S)\) be a non-empty set, and \((\star)\) said to be a binary operation on \((S)\), if \((a \star b)\) is defined for all \((a, b \in S)\). In other words, \((\star)\) is a rule for any two elements in the set \((S)\).

**Example \((PageIndex{1})\): Binary operations**

The following are binary operations on \((\mathbb{Z})\):

1. The arithmetic operations, addition \((+ )\), subtraction \((- )\), multiplication \((\times )\), and division \((/ )\).
2. Define an operation oplus on \((\mathbb{Z})\) by \((a \oplus b = ab + a + b, \forall a, b \in \mathbb{Z})\).
3. Define an operation ominus on \((\mathbb{Z})\) by \((a \ominus b = ab + a - b, \forall a, b \in \mathbb{Z})\).
4. Define an operation otimes on \((\mathbb{Z})\) by \((a \otimes b = (a+b)(a+b), \forall a, b \in \mathbb{Z})\).
5. Define an operation oslash on \((\mathbb{Z})\) by \((a \oslash b = (a+b)(a-b), \forall a, b \in \mathbb{Z})\).
6. Define an operation min on \((\mathbb{Z})\) by \((a \vee b = \min \{a, b\}, \forall a, b \in \mathbb{Z})\).
7. Define an operation max on \((\mathbb{Z})\) by \((a \wedge b = \max \{a, b\}, \forall a, b \in \mathbb{Z})\).
8. Define an operation defect on \(\mathbb{Z}\) by \(a \ast_3 b = a+b-3, \forall a,b \in \mathbb{Z}\).

Let's explore the binary operations, before we proceed:

Example \(\PageIndex{2}\):
1. \((2 \oplus 3)=(2)(3)+2+3=11\).
2. \((2 \otimes 3)=(2+3)(2+3)=25\).
3. \((2 \oslash 3)=(2+3)(2-3)=-5\).
4. \((2 \ominus 3)=(2)(3)+2-3=5\).
5. \((2 \vee 3 = 2)\).
6. \((2 \wedge 3 = 3)\).

Exercise \(\PageIndex{2}\):
1. \((-2 \oplus 3)\).
2. \((-2 \otimes 3)\).
3. \((-2 \oslash 3)\).
4. \((-2 \ominus 3)\).
5. \((-2 \vee 3)\).
6. \((-2 \wedge 3)\).

Answer

\(-5, 1, 5, -2, 3\)

Properties:

Closure property

Definition: Closure property

Let \(\langle S \rangle\) be a non-empty set. A binary operation \(\langle \star \rangle\) on \(\langle S \rangle\) is said to be a closed binary operation on \(\langle S \rangle\), if \(\langle a \star b \in S, \forall a, b \in \langle S \rangle \rangle\).

Below we shall give some examples of closed binary operations, that will be further explored in class.
Example \(\PageIndex{3}\): Closed binary operations

The following are closed binary operations on \(\mathbb{Z}\).

1. The addition \(+\), subtraction \(-\), and multiplication \(\times\).
2. Define an operation oplus on \(\mathbb{Z}\) by \(a \oplus b = ab + a + b, \forall a, b \in \mathbb{Z}\).
3. Define an operation ominus on \(\mathbb{Z}\) by \(a \ominus b = ab + a - b, \forall a, b \in \mathbb{Z}\).
4. Define an operation otimes on \(\mathbb{Z}\) by \(a \otimes b = (a+b)(a+b), \forall a, b \in \mathbb{Z}\).
5. Define an operation oslash on \(\mathbb{Z}\) by \(a \oslash b = (a+b)(a-b), \forall a, b \in \mathbb{Z}\).
6. Define an operation min on \(\mathbb{Z}\) by \(a \vee b = \min \{a, b\}, \forall a, b \in \mathbb{Z}\).
7. Define an operation max on \(\mathbb{Z}\) by \(a \wedge b = \max \{a, b\}, \forall a, b \in \mathbb{Z}\).
8. Define an operation defect on \(\mathbb{Z}\) by \(a \ast_3 b = a + b - 3, \forall a, b \in \mathbb{Z}\).

Exercise \(\PageIndex{1}\)

Determine whether the operation ominus on \(\mathbb{Z_+}\) is closed?

**Answer**

The operation ominus on \(\mathbb{Z_+}\) is closed.

Example \(\PageIndex{4}\): Counter Example

Division \(\div\) is not a closed binary operations on \(\mathbb{Z}\).

\(2, 3 \in \mathbb{Z}\) but \(\frac{2}{3} \notin \mathbb{Z}\).

Summary of arithmetic operations and corresponding sets:

<table>
<thead>
<tr>
<th>Operation</th>
<th>(\mathbb{Z_+})</th>
<th>(\mathbb{Z})</th>
<th>(\mathbb{Q})</th>
<th>(\mathbb{R})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(+)</td>
<td>closed</td>
<td>closed</td>
<td>not closed</td>
<td>not closed</td>
</tr>
<tr>
<td>(\times)</td>
<td>closed</td>
<td>closed</td>
<td>closed</td>
<td>closed</td>
</tr>
<tr>
<td>(-)</td>
<td>not closed</td>
<td>closed</td>
<td>closed</td>
<td>closed</td>
</tr>
<tr>
<td>(\div)</td>
<td>not closed</td>
<td>not closed</td>
<td>closed (only when (0) is not included)</td>
<td>closed (only when (0) is not included)</td>
</tr>
</tbody>
</table>
**Associative property**

**Definition: Associative property**

Let \( S \) be a subset of \( \mathbb{Z} \). A binary operation \( \star \) on \( S \) is said to be associative, if \( (a \star b) \star c = a \star (b \star c) \), \( \forall a, b, c \in S \).

We shall assume the fact that the addition \((+)\) and the multiplication \((\times)\) are associative on \( \mathbb{Z}_+ \). (You don't need to prove them!)

Below is an example of proof when the statement is True.

**Example \( \PageIndex{5} \): Associative**

Determine whether the binary operation oplus is associative on \( \mathbb{Z} \).

We shall show that the binary operation oplus is associative on \( \mathbb{Z} \).

**Proof:**

Let \( (a,b,c) \in \mathbb{Z} \). Then consider, \( ((a \oplus b) \oplus c = (ab+a+b) \oplus c = (ab+a+b)c+(ab+a+b)+c = (ab)c+ac+bc+ab+a+b+c) \).

On the other hand, \( (a \oplus (b \oplus c) =a \oplus (bc+b+c)=a(bc+b+c)+a+(bc+b+c)=a(bc)+ab+ac+a+bc+b+c) \)

Since multiplication is associative on \( \mathbb{Z} \), \( (a \oplus (b \oplus c =a \oplus (b \oplus c) ) \)

Thus, the binary operation oplus is associative on \( \mathbb{Z} \). \( \Box \)

Below is an example of how to disprove when a statement is False.

**Example \( \PageIndex{6} \): Not Associative**

Determine whether the binary operation subtraction \((\cdot-\cdot)\) is associative on \( \mathbb{Z} \).
Answer: The binary operation subtraction \((-\)) is not associative on \(\mathbb{Z}\).

**Counter Example:**

Choose \((a=2, b=3, c=4, l)\) then \((2-3)-4=-1-4=-5\), but \((2-(3-4))=2-(-1)=2+1=3\).

Hence the binary operation subtraction \((-\)) is not associative on \(\mathbb{Z}\).

---

**Commutative property**

**Definition: Commutative property**

Let \(\text{S}\) be a non-empty set. A binary operation \(\star\) on \(\text{S}\) is said to be commutative, if \(a \star b = b \star a, \forall a, b \in \text{S}\).

We shall assume the fact that the addition \((+\)) and the multiplication \((\times)\) are commutative on \(\mathbb{Z_+}\). *(You don’t need to prove them!)*

Below is the proof of subtraction \((-\)) NOT being commutative.

**Example \(\PageIndex{7}\): NOT Commutative**

Determine whether the binary operation subtraction \((-\)) is commutative on \(\mathbb{Z}\).

**Counter Example:**

Choose \(a=3\) and \(b=4\).

Then \((a-b=3-4=-1)\), and \((b-a= 4-3=1)\).

Hence the binary operation subtraction \((-\)) is not commutative on \(\mathbb{Z}\).

---

**Example \(\PageIndex{8}\): Commutative**

Determine whether the binary operation oplus is commutative on \(\mathbb{Z}\).

We shall show that the binary operation oplus is commutative on \(\mathbb{Z}\).

**Proof:**

Let \((a, b) \in \mathbb{Z}\).
Then consider, \((a \oplus b) = (ab+a+b).\)

On the other hand, \((b \oplus a) = ba+b+a.\)

Since multiplication is associative on \(\mathbb{Z}\), \((a \oplus b) = (b \oplus a).\)

Thus, the binary operation \(\oplus\) is commutative on \(\mathbb{Z}\). \(\Box\)

---

**Identity**

**Definition: Identity**

A non-empty set \(S\) with binary operation \(\star\), is said to have an identity \((e \in S), if \((e \star a=a \star e=a), \forall a \in S.\)

Note that \((0)\) is called additive identity on \((\mathbb{Z}, +)\), and \((1)\) is called multiplicative identity on \((\mathbb{Z}, \times)\).

**ExamplePageIndex(9): Is identity unique?**

Let \((S)\) be a non-empty set and let \((\star)\) be a binary operation on \((S)\). If \((e_1)\) and \((e_2)\) are two identities in \((S, \star)\), then \((e_1=e_2)\).

**Proof:**

Suppose that \((e_1)\) and \((e_2)\) are two identities in \((S, \star)\).

Then \((e_1=e_1 \star e_2=e_2)\)

Hence identity is unique. \(\Box\)

**ExamplePageIndex(10): Identity**

Does \((\mathbb{Z}, \oplus)\) have an identity?

**Answer:**

Let \((e)\) be the identity on \((\mathbb{Z}, \oplus))\).

Then \((e \oplus a=a \oplus e=a), \forall a \in \mathbb{Z}.)\)

Thus \((ae+a+e=a), and (ae+a+e=a) \forall a \in \mathbb{Z}.)\)

---

UC Davis ChemWiki is licensed under a Creative Commons Attribution-Noncommercial-Share Alike 3.0 United States License.
Since \((ea+e+a=a)\) \(\forall a \in \mathbb{Z}\), \((ea+e=0 \implies e(a+1)=0)\) \(\forall a \in \mathbb{Z}\).

Therefore \((e=0)\).

Now \(0 \oplus a=a \oplus 0=a, \forall a \in \mathbb{Z}\).

Hence \(0\) is the identity on \(\langle \mathbb{Z}, \oplus \rangle\).

---

**Example \((PageIndex{11})\):**

Does \(\langle \mathbb{Z}, \otimes \rangle\) have an identity?

**Answer:**

Let \((e)\) be the identity on \(\langle \mathbb{Z}, \otimes \rangle\).

Then \((e \otimes a = a \otimes e = a)\) \(\forall a \in \mathbb{Z}\).

Thus \((e(a+e)=(a+e)(a+e)=a)\) \(\forall a \in \mathbb{Z}\).

Now, \((a+e)(a+e)=a, \forall a \in \mathbb{Z}\).

\((\implies a^2+2ea+e^2=a)\) \(\forall a \in \mathbb{Z}\).

Choose \((a=0)\) then \((e=0)\).

If \((e=0)\) then \((a^2=a)\) \(\forall a \in \mathbb{Z}\).

This is a contradiction.

Hence, \(\langle \mathbb{Z}, \otimes \rangle\) has no identity.

---

**Distributive Property**

**Definition: Distributive property**

Let \((S)\) be a non-empty set. Let \((\star_1)\) and \((\star_2)\) be two different binary operations on \((S)\).

Then \((\star_1)\) is said to be distributive over \((\star_2)\) on \((S)\) if \((a \star_1 (b \star_2 c) = (a \star_1 b) \star_2 (a \star_1 c), \\forall a,b,c \in S)\).
Note that the multiplication distributes over the addition on $\mathbb{Z}$. That is, $4(10+6)=(4)(10)+(4)(6)=40+24=64$.

Further, we extend to $(a+b)(c+d) = (ac+ad+bc+bd)$ (FOIL).

F-First
O-Outer
I-Inner
L-Last

This property is very useful to find $(26)(27)$ as shown below:

Example (PageIndex{12}): Find $(26)(27)$

\[
\begin{array}{ccc}
20 & 6 \\
20 & 400 & 120 \\
7 & 140 & 42 \\
\end{array}
\]

Hence $(26)(27)=400+120+140+42=702$.

Let's play a game!

Example (PageIndex{13})

Does multiplication distribute over subtraction?

Example (PageIndex{14})

Does division distribute over addition?

Answer:

Counter Example:

Choose $a = 2$, $b = 3$, $c = 4$.

Then $a \div (b + c) = 2 \div (3+4)$.
= \(\frac{2}{7}\).

and \((a \div b) + (a \div c) = \left(\frac{2}{7}\right) + \left(\frac{2}{4}\right)\).

= \(\frac{7}{6}\).

Since \(\frac{2}{7} \neq \frac{7}{6}\), the binary operation \(\div\) is not distributive over +.

**Example \(\PageIndex{15}\):**

Does \(\otimes\) distribute over \(\oplus\) on \(\mathbb{Z}\)?

**Answer:**

**Counter Example:**

Choose \(a = 2, b = 3, c = 4\).

Then

\[
2\otimes(3\oplus4) = 2\otimes((3\cdot4)+3+4)
\]

\[
= 2\otimes19
\]

\[
= (2+19)(2+19)
\]

\[
= 441
\]

and

\[
(2\otimes3)\oplus(2\otimes4) = (2+3)(2+4)\oplus(2+4)(2+4)
\]

\[
= 25\oplus36
\]

\[
= (25)(36)+25+36
\]

\[
= 961.
\]

Since \(441 \neq 961\), the binary operation \(\otimes\) is not distributive over \(\oplus\) on \(\mathbb{Z}\).

**Summary**

In this section, we have learned the following for a non-empty set \(S\):

1. Binary operation,
2. Closure property,
3. Associative property,
4. Commutative property,
5. Distributive property, and
6. Identity.