6.5: Varieties

As before, if \( f \colon U \to \mathbb{C} \) is a function, let \( Z_f = f^{-1}(0) \subset U \) denote the zero set of \( f \).

**Definition: Complex**

Let \( U \subset \mathbb{C}^n \) be an open set. Let \( X \subset U \) be a set such that near each point \( (p \in U) \), there exists a neighborhood \( (W) \) of \( (p) \) and a family of holomorphic functions \( \mathcal{F} \) defined on \( (W) \) such that \( (W \cap X = \{ z \in W : f(z) = 0 \text{ for all } f \in \mathcal{F} \}) = \bigcap_{f \in \mathcal{F}} Z_f \). Then \( (X) \) is called a (complex or complex-analytic) variety or a subvariety of \( (U) \). Sometimes \( (X) \) is called an analytic set. We say \( (X \subset U) \) is a proper subvariety if \( (\emptyset \not= X \subsetneq U) \).

We generally leave out the “complex” from “complex subvariety” as it is clear from context. But you should know that there are other types of subvarieties, namely real subvarieties given by real-analytic functions. We will not cover those in this book.

**Example \( \PageIndex{1} \)**

The set \( (X = \{ 0 \}) \subset \mathbb{C}^n \) is a subvariety as it is the only common vanishing point of functions \( \{ z_1, \ldots, z_n \} \). Similarly, \( (X = \mathbb{C}^n) \) is a subvariety of \( (\mathbb{C}^n) \), where we let \( \{ z_1, \ldots, z_n \} \).

**Example \( \PageIndex{2} \)**

The set defined by \( (z_2 = e^{1/z_1}) \) is a subvariety of \( (U = \mathbb{C}^2 : z_1 \not=0) \). It is not a
subvariety of any open set larger than \( \langle U \rangle \).

It is useful to note what happens when we replace “near each point \( \langle p \in U \rangle \)” with “near each point \( \langle p \in X \rangle \).” We get a slightly different concept, and \( \langle X \rangle \) is said to be a local variety. A local variety \( \langle X \rangle \) is a subvariety of some neighborhood of \( \langle X \rangle \), but it is not necessarily closed in \( \langle U \rangle \). As a simple example, the set \( \langle X = \{ z \in \mathbb{C}^2 \mid z_1 = 0, |z_2| < 1 \} \rangle \) is a local variety, but not a subvariety of \( \langle \mathbb{C}^2 \rangle \). On the other hand, \( \langle X \rangle \) is a subvariety of the unit ball \( \langle \{ z \in \mathbb{C}^2 \mid \|z\| < 1 \} \rangle \).

Note that \( \langle \text{mathcal{F}} \rangle \) depends on \( \langle p \rangle \) and near each point may have a different set of functions. Clearly the family \( \langle \text{mathcal{F}} \rangle \) is not unique. We will prove below that we would obtain the same definition if we restricted to finite families \( \langle \text{mathcal{F}} \rangle \).

We work with germs of functions. Recall, that when \( \langle (f,p) \rangle \) is a germ of a function the germ \( \langle (Z_f,p) \rangle \) is the germ of the zero set of some representative. Let \( \langle I_p(X) \overset{\text{def}}{=} \{ (f,p) \in \mathcal{O}_p : (X,p) \subset (Z_f,p) \} \rangle \). That is, \( \langle I_p(X) \rangle \) is the set of germs of holomorphic functions vanishing on \( \langle X \rangle \) near \( \langle p \rangle \). If a function vanishes on \( \langle X \rangle \), then any multiple of it also vanishes on \( \langle X \rangle \), so \( \langle I_p(X) \rangle \) is an ideal. Really \( \langle I_p(X) \rangle \) depends only on the germ of \( \langle X \rangle \) at \( \langle p \rangle \), so define \( \langle I_p((X,p)) \rangle = \langle I_p(X) \rangle \).

Every ideal in \( \langle \text{mathcal{O}}_p \rangle \) is finitely generated. Let \( \langle I \subset \mathcal{O}_p \rangle \) be an ideal generated by \( \langle f_1, f_2, \ldots, f_k \rangle \). Write \( \langle V(I) \overset{\text{def}}{=} \{ (Z_{f_1},p) \cap (Z_{f_2},p) \cap \cdots \cap (Z_{f_k},p) \} \rangle \). That is, \( \langle V(I) \rangle \) is the germ of the subvariety “cut out” by the elements of \( \langle I \rangle \), since every element of \( \langle I \rangle \) vanishes on the points where all the generators vanish. Suppose representatives \( \langle f_1, f_2, \ldots, f_k \rangle \) of the generators are defined in some neighborhood \( \langle W \rangle \) of \( \langle p \rangle \), and a germ \( \langle (g,p) \in \mathcal{O}_p \rangle \) has a representative \( \langle g(p) \rangle \) defined in \( \langle W \rangle \) such that \( \langle g = c_1 f_1 + \cdots + c_k f_k \rangle \), where \( \langle c_k \rangle \) are also holomorphic functions on \( \langle W \rangle \). If \( \langle q \in Z_{f_1} \cap \cdots \cap Z_{f_k} \rangle \), then \( \langle g(q) = 0 \rangle \). Thus, \( \langle Z_{f_1} \cap \cdots \cap Z_{f_k} \rangle \subset Z_g \), or in terms of germs, \( \langle V(I) \subset (Z_g,p) \rangle \). The reason why we did not define \( \langle V(I) \rangle \) to be the intersection of zero sets of all germs in \( \langle I \rangle \) is that this would be an infinite intersection, and we did not define such an object for germs.

**Exercise \( \langle \text{PageIndex(1)} \rangle \)**

Show that \( \langle V(I) \rangle \) is independent of the choice of generators.

**Exercise \( \langle \text{PageIndex(2)} \rangle \)**

Suppose \( \langle I_p(X) \rangle \) is generated by the functions \( \langle f_1, f_2, \ldots, f_k \rangle \). Prove \( \langle (X,p) = (Z_{f_1},p) \cap (Z_{f_2},p) \cap \cdots \cap (Z_{f_k},p) \rangle \).

**Exercise \( \langle \text{PageIndex(3)} \rangle \)**

Given a germ \( \langle (X,p) \rangle \) of a subvariety at \( \langle p \rangle \), show \( \langle V(\text{bigl}(I_p(X)\text{bigr})) = (X,p) \rangle \) (see above). Then given an ideal \( \langle I \subset \text{mathcal{O}}_p \rangle \), show \( \langle I_p(V(I)) \rangle \).

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As \((\mathcal{O}_p)\) is Noetherian, \((I_p(X))\) is finitely generated. Near each point \((p)\) only finitely many functions are necessary to define a subvariety, that is, by an exercise above, those functions “cut out” the subvariety. When one says \textit{defining functions} for a germ of a subvariety, one generally means that those functions generate the ideal, not just that their common zero set happens to be the subvariety. A theorem that we will not prove here in full generality, the \textit{Nullstellensatz}, says that if we take the germ of a subvariety defined by functions in an ideal \((I \subset \mathcal{O}_p)\), and look at the ideal given by that subvariety, we obtain the radical \((\sqrt{I})\) of \((I)\). In more concise language, the Nullstellensatz says \((I_p \bigl(V(I)\bigr) = \sqrt{I})\). Germs of subvarieties are in one-to-one correspondence with radical ideals of \((\mathcal{O}_p)\).

\textbf{Example \((\PageIndex{3})\)}

The subvariety \((X = \{0\} \subset \mathbb{C}^2)\) can be given by \((\mathcal{F} = \{z_1^2, z_2^2\})\). If \((I = \text{bigl}(z_1^2, z_2^2\bigr) \subset \mathcal{O}_0)\) is the ideal of germs generated by these two functions, then \((I_0(X) \neq I)\). We have seen that the ideal \((I_0(X))\) is the maximal ideal \((\mathfrak{m}_0 = (z_1, z_2))\). If we prove that all the nonconstant monomials are in \((\sqrt{I})\), then \((\sqrt{I}) = (z_1, z_2) = \mathfrak{m}_0\). The only nonconstant monomials that are not in \((I)\) are \((z_1), (z_2),\) and \((z_1z_2)\), but the square of each of these is in \((I)\), so \((\sqrt{I} = \mathfrak{m}_0)\).

The local properties of a subvariety at \((p)\) are encoded in the properties of the ideal \((I_p(X))\). Therefore, the study of subvarieties often involves the study of the various algebraic properties of the ideals of \((\mathcal{O}_p)\). Let us also mention in passing that the other object that is studied is the so-called \textit{coordinate ring} \((\mathcal{O}_p / I_p(X))\), which represents the functions on \((X,p))\). That is, we identify two functions if they differ by something in the ideal, since then they are equal on \((X)\).

At most points a subvariety behaves like a piece of \((\mathbb{C}^k)\), more precisely like a graph over \((\mathbb{C}^k)\). A graph of a mapping \(f \colon U' \subset \mathbb{C}^k \to \mathbb{C}^{n-k}\) is the set \(\Gamma_f \subset U' \times \mathbb{C}^{n-k} \subset \mathbb{C}^k \times \mathbb{C}^{n-k}\) defined by \(\Gamma_f \overset{\text{def}}{=} \{ (z, w) \in U' \times \mathbb{C}^{n-k} : w = f(z) \}.\)

\textbf{Definition: Complex Manifold}

Let \((X \subset U \subset \mathbb{C}^n)\) be a subvariety of an open set \((U)\). Let \((p \in X)\) be a point. Suppose that after a permutation of coordinates, near \((p)\) the set \((X)\) is a graph of a holomorphic mapping. That is, after relabeling coordinates, there is a neighborhood \((U' \times \mathbb{C}^{n-k}) \subset \mathbb{C}^k \times \mathbb{C}^{n-k})\) of \((p)\) such that \((X \cap (U' \times \mathbb{C}^{n-k})) = \Gamma_f\) for a holomorphic mapping \(f \colon U' \to \mathbb{C}^{n-k}\). Then \((p)\) is a \textit{regular point} (or \textit{simple point}) of \((X)\) and the \textit{dimension} of \((X)\) at \((p)\) is \((k)\). We write \((\dim_p X = k)\). If all points of \((X)\) are regular points of dimension \((k)\), then \((X)\) is called a \textit{complex manifold}, or \textit{complex submanifold}, of (complex) dimension \((k)\).

As the ambient \((\mathbb{C}^2)\) dimension is \((n)\), we say \((X)\) is of \textit{codimension} \((n-k)\) at \((p)\).

The set of regular points of \((X)\) is denoted by \((X_{\math{\text{reg}}} = \{ x \in X : x \text{ is regular} \})\). Any point that is not regular is \textit{singular}. The set of
singular points of \( \mathcal{X} \) is denoted by \( \mathcal{X}_{\text{sing}} \).

A couple of remarks are in order. A subvariety \( \mathcal{X} \) can have regular points of several different dimensions, although if a point is a regular point of dimension \( k \), then all nearby points are regular points of dimension \( k \) as the same \( \mathcal{U} \) and \( \mathcal{U}^{\prime} \) works. Any isolated point of \( \mathcal{X} \) is automatically a regular point of dimension 0. Finally, remark is that dimension is well-defined. We leave it as an exercise. Sometimes the empty set is considered a complex manifold of dimension \(-1\) (or \(-\infty\)).

**Example \( \PageIndex{4} \)**

The set \( \mathcal{X} = \mathbb{C}^n \) is a complex submanifold of dimension \( n \) (codimension 0). In particular, \( \mathcal{X}_{\text{reg}} = \mathcal{X} \) and \( \mathcal{X}_{\text{sing}} = \emptyset \).

The set \( \mathcal{Y} = \{ z \in \mathbb{C}^3 : z_3 = z_1^2 - z_2^2 \} \) is a complex submanifold of dimension \( 2 \) (codimension 1). Again, \( \mathcal{Y}_{\text{reg}} = \mathcal{Y} \) and \( \mathcal{Y}_{\text{sing}} = \emptyset \).

On the other hand, the so-called *cusp*, \( \mathcal{C} = \{ z \in \mathbb{C}^2 : z_1^3 - z_2^2 = 0 \} \) is not a complex submanifold. The origin is a singular point of \( \mathcal{C} \) (see exercise below). At every other point we can write \( z_2 = \pm z_1^{3/2} \), so \( \mathcal{C}_{\text{reg}} = \mathcal{C} \setminus \{0\} \), and so \( \mathcal{C}_{\text{sing}} = \{ 0 \} \). The dimension at every regular point is 1. See for a plot of \( \mathcal{C} \) in two real dimensions:

![Plot of C](image)

**Exercise \( \PageIndex{4} \)**

Prove that if \( \mathcal{X} \) is a regular point of a subvariety \( \mathcal{X} \subset \mathcal{U} \subset \mathbb{C}^n \) of a domain \( \mathcal{U} \), then the dimension at \( \mathcal{X} \) is well-defined. Hint: If there were two possible \( \mathcal{U}^{\prime} \) of different dimension (possibly different affine coordinates), construct a map from one such \( \mathcal{U}^{\prime} \) to another such \( \mathcal{U}^{\prime} \) with nonvanishing derivative.
Exercise \(\PageIndex{5}\))

Consider the cusp \((C = \{ z \in \mathbb{C}^2 : z_1^3-z_2^2 = 0 \})\). Prove that the origin is not a regular point of \(C\).

Exercise \(\PageIndex{6}\))

Show that \(p\) is a regular point of dimension \(k\) of a subvariety \(X\) if and only if there exists a local biholomorphic change of coordinates that puts \(p\) to the origin and near \(0\), \(X\) is given by \(w=0\), where \((z,w) \in \mathbb{C}^{k} \times \mathbb{C}^{n-k}\). In other words, if we allow a biholomorphic change of coordinates, we can let \(f=0\) in the definition.

We also define dimension at a singular point. A fact that we will not prove in general is that the set of regular points of a subvariety is open and dense in the subvariety; a subvariety is regular at most points. Therefore, the following definition makes sense without resorting to the convention that \(\max \emptyset = \infty\).

**Definition: Dimension**

Let \(X \subset U \subset \mathbb{C}^n\) be a (complex) subvariety of \(U\). Let \((p \in X)\) be a point. We define the dimension of \(X\) at \((p)\) to be \(\dim_p X = \max \{ \dim_{n-1} X_{\text{reg}} \} \) with \(\dim_{n-1} X_{\text{reg}} \) being the dimension of \(X\) at \((p)\). The dimension of the entire subvariety \(X\) is defined to be \(\dim X = \max \{ \dim_{n-1} X_{\text{reg}} \} \) with \(\dim_{n-1} X_{\text{reg}} \) being the dimension of \(X\) at \((p)\). We say that \(X\) is of pure dimension \(k\) at \((p)\) if at all points \((p)\), dimension of \(X\) at \((p)\) is \(k\). We say a germ \((X,p)\) is of pure dimension \(k\) if there exists a representative of \(X\) that is of pure dimension \(k\). We define the word codimension as before, that is, the ambient dimension minus the dimension of \(X\).

**Example \(\PageIndex{5}\))**

We saw that \((C = \{ z \in \mathbb{C}^2 : z_1^3-z_2^2 = 0 \})\) is of dimension 1 at all the regular points, and the only singular point is the origin. Hence \((\dim C = 1)\), and so \(\dim C = 1\). The subvariety \(C\) is of pure dimension 1.

We have the following theorem, which we state without proof, at least in the general setting.

**Theorem \(\PageIndex{1}\))**

Let \((U \subset \mathbb{C}^n)\) be open and connected and let \((X \subset U)\) be a subvariety, then the set of regular points \((X_{\text{reg}})\) is open and dense in \((X)\). In fact, \((X_{\text{sing}})\) is a subvariety.
Exercise \(\PageIndex{7}\))

Suppose that \(X \subset U \subset \mathbb{C}^n\) is a subvariety of a domain \((U)\), such that \((X_{\text{reg}})\) is connected. Show that \(X\) is of pure dimension. Feel free to assume \((X_{\text{reg}})\) is dense in \(X\).

Footnotes

[1] The radical of \(I\) is defined as \(\sqrt{I} \overset{\text{def}}{=} \{ f : f^m \in I, \text{ for some } m \}\).

[2] The word ambient is used often to mean the set that contains whatever object we are talking about.