2.2: Equivalence Relations, and Partial order

Definition

A binary relation is an equivalence relation on a non-empty set \( \langle S \rangle \) if and only if the relation is reflexive(R), symmetric(S) and transitive(T).

Definition

A binary relation is a **partial order** if and only if the relation is reflexive(R), antisymmetric(A) and transitive(T).

**Example \( \langle \text{PageIndex\{1\}} \rangle : \langle (=) \rangle **

Let

\[ S = \mathbb{R} \]

and

\[ R \]

be \( = \). Is the relation a) reflexive, b) symmetric, c) antisymmetric, d) transitive, e) an equivalence relation, f) a partial order.

**Solution:**

1. Yes

\[ R \]
is reflexive.

**Proof:**

Let
\[ a \in \mathbb{R} \,.
\]

Then
\[ a = a \,.
\]

Hence
\[ R \]

is reflexive.

2. Yes

\[ R \]

is reflexive.

**Proof:**

Let
\[ (a, b) \in \mathbb{R} \,.
\]

If
\[ a = b \]

, clearly
\[ b = a \,.
\]

Hence
\[ R \]

is symmetric.

1. Yes

\[ R \]

is antisymmetric.

**Proof:**
Let 
\( a, b \in \mathbb{R} \)
s.t.
\[ a = b \]
and
\[ b = a \]
then clearly
\[ a = b \quad \forall a, b \in \mathbb{R} \]

2. Yes
\( R \)
is transitive.

**Proof:**

Let 
\( a, b, c \in \mathbb{R} \)
s.t.
\[ a = b \]
and
\[ b = c \]

We shall show that
\( aRc \)

Since
\[ a = b \]
and

it follows that
\[ a = c \]

Thus
\( aRc \)

?
3. Yes

\( R \)
is an equivalence relation.

**Proof:**

Since

\( R \)
is reflexive, symmetric and transitive, it is an equivalence relation.

4. Yes

\( R \)
is a partial order.

**Proof:**

\( R \)
is a partial order, since

\( R \)
is reflexive, antisymmetric and transitive.

**Example \( \PageIndex{2} \): Less than or equal to**

Let

\( S = \mathbb{R} \)
and

\( R \)
be

\( \leq \).

Is the relation a) reflexive, b) symmetric, c) antisymmetric, d) transitive, e) an equivalence relation, f) a partial order.

**Solution**

1. Yes,

\( R \)
is reflexive.

**Proof:**

We will show that

\( a R a \)
is true.
Let
\( a \in S \)
that is
\( a \in \mathbb{R} \).

Since
\( a \leq a \),
\( R \)
is reflexive.

2. No,
\[ R \]
is not symmetric.

Counterexample:

Let
\( a = 2 \)
and
\( b = 3 \)
which are both
\( \in \mathbb{R} \).

It is true that
\( 2 \leq 3 \), but it is not true that
\( 3 \leq 2 \).

Thus
\( R \)
is not symmetric.

3. Yes,
\( R \)
is antisymmetric.
Proof:

We will show that given

\[ a \leq b \]
and

\[ b \leq a \]
that

\[ a = b \]

Since

\[ a \leq b \]
,

\[ \exists \{ x \in \mathbb{R} : x \geq 0 \} \]
s.t.

\[ a + x = b \]

Further, since

\[ b \leq a \]
,

\[ \exists \{ y \in \mathbb{R} : y \geq 0 \} \]
,

\[ b + y = a \]

Then

\[ a + x = a - y \]

Thus,

\[ x + y = 0 \]

Since,

\[ x, y \geq 0 \]
,

\[ x, y = 0 \]
thus

\[ a = b \]
4. Yes, $R$ is transitive.

**Proof:**

We will show that given

\[ a \leq b \]

and

\[ b \leq c \]

that

\[ a \leq c \]

Since

\[ a \leq b \]

\[ \exists \{ x \in \mathbb{R} : x \geq 0 \} \]

s.t.

\[ a + x = b \]

Further, since

\[ b \leq c \]

\[ \exists \{ y \in \mathbb{R} : y \geq 0 \} \]

\[ b + y = c \]

Then

\[ a + x + y \leq c \]

Since,

\[ x, y \geq 0 \]

thus

\[ a \leq c \]
5. No,
\( R \)
is not an equivalence relation on
\( S \)
since it is not symmetric.

6. Yes,
\( R \)
is a partial order on
\( S \)
since it is reflexive, antisymmetric and transitive.

Definition

Given an equivalence relation \( (R) \) over a set \( (S, \) \) for any \( (a \in S) \) the equivalence class of a is the set \( ([a]_R =\{ b \in S \mid a \ R \ b \}) \), that is
\( ([a]_R) \) is the set of all elements of \( S \) that are related to \( (a) \).

**Example \( (PageIndex(3)) \): Equivalence relation**

Define a relation that two shapes are related iff they are the same color. Is this relation an equivalence relation?

Equivalence classes are:
Example \(\PageIndex{4}\):

Define a relation that two shapes are related iff they are similar. Is this relation an equivalence relation?

Equivalence classes are:

![Equivalence classes image]

Theorem \(\PageIndex{1}\)

If \(\sim\) is an equivalence relation over a non-empty set \((S)\). Then the set of all equivalence classes is denoted by \(\{[a]_{\sim}\mid a \in S\}\) forms a partition of \((S)\).

This means

1. Either \([a] \cap [b] = \emptyset\) or \([a] = [b]\), for all \(a, b \in S\).
2. \((S = \cup_{a \in S} [a])\).

Proof

Assume

\(\sim\)

is an equivalence relation on a non-empty set

\(S\).

Let

\(x, y \in S\).

If
\[ [x] \sim \cap [y] \sim = \emptyset \]
then we are done. Otherwise,

assume
\[ [x] \sim \cap [y] \sim \neq \emptyset \]

Let
\[ a \]
be the common element between them.

Let
\[ a \in [x] \sim \cap [y] \sim \]

Then
\[ a \in [x] \sim \]

and
\[ a \in [y] \sim \]

, which means that
\[ a \sim x \]

and
\[ a \sim y \]

Since
\[ \sim \]
is an equivalence relation and
\[ a \sim x, x \sim a \]

Since
\[ x \sim a \]

and
\[ a \sim y \]

(due to transitive property),
\[ x \sim y \]

Thus
\[ y \in [x] \sim \]

and

-------------------------------------------------------------------------------------------------------------------------------------
\( x \in [y]_\sim \).

Hence
\( [x]_\sim = [y]_\sim \).

Next, we will show that
\( S = \bigcup_{x \in S} [x]_\sim \).

First we shall show that
\( S \subseteq \bigcup_{x \in S} [x]_\sim \).

Let
\( m \in S \).

Then
\( [m]_\sim \) exists and
\( m \in [m]_\sim \).

Hence
\( m \in \bigcup_{x \in S} [x]_\sim \).

Thus
\( S \subseteq \bigcup_{x \in S} [x]_\sim \).

Conversely, we shall show that
\( \bigcup_{x \in S} [x]_\sim \subseteq S \).

Let
\[ d \in \bigcup_{x \in S} [x]_\sim \]

Then
\[ d \in [x]_\sim \]
for some
\[ x \in S \]

Thus
\[ d \sim x \]
\[ d \in x \]

Thus
\[ \bigcup_{x \in S} [x]_\sim \subseteq S \]

Since
\[ S \subseteq \bigcup_{x \in S} [x]_\sim \]
and
\[ \bigcup_{x \in S} [x]_\sim \subseteq S \]
then
\[ S = \bigcup_{x \in S} [x]_\sim \]

Note
For every equivalence relations over a nonempty set \( \setminus (S) \), \( \setminus (S) \) has a partition.

For the following examples, determine whether or not each of the following binary relations
\[ R \]
on the given set
\[ A \]
is reflexive, symmetric, antisymmetric, or transitive. If a relation has a certain property, prove this is so; otherwise, provide a counterexample to show that it does not. If
\[ R \]
is an equivalence relation, describe the equivalence classes of $A$.

Example ((PageIndex(5))))

Let $A = S \times S$.

. Define a relation $R$ on $A$ by

$(a, b) R (c, d)$ if and only if

$10a + b \leq 10c + d$.

Solution

1. $R$ is reflexive on $S$.

Proof:

Let $(a, b) \in A$.

We will show that

$10a + b \leq 10a + b$.

Since

$10a + b = 10a + b$,

then

$10a + b \leq 10a + b$. 
Since

\((a, b) R (a, b)\)

\(R\)
is reflexive on

\(S\).

2.

\(R\)
is not symmetric on

\(S\).

**Counter Example:**

Let

\(a = 0, b = 1, c = 2, d = 3\).

Note

\(10(0) + 1 \leq 10(2) + 3\), specifically

\((a, b) R (c, d)\),

\(1 \leq 23\)
is true.

However,

\((c, d) R (a, b)\),

\(23 \leq 1\)
is false.

Since

\(b \not{R} a\),

\(R\)
is not symmetric on

\(S\).
3. 

\( R \)

is antisymmetric on 

\( S \).

**Proof:**

Let 

\[ a, b, c, d \in S \]

s.t.

\[ 10a + b \leq 10c + d \]

and

\[ 10c + d \leq 10a + b \].

Since

\[ 10a + b \leq 10c + d \]

and

\[ 10c + d \leq 10a + b \],

\[ 10a + b = 10c + d \].

We will show that

\[ a = c \]

and

\[ b = d \].

\[ 0 \leq a, b, c, d \leq 9 \]

by definition.

Since

\[ 0 \leq b, d \leq 9 \]

and

\[ 0 \leq a, c \leq 9 \],
Thus
\[(a, b)R(c, d)\]
is antisymmetric on
\[S\]?

5.
\[R\]
is transitive on
\[S\].

**Proof:**

Let
\[a, b, c, d, e, f \in S\]
s.t.
\[10a + b \leq 10c + d\]
and
\[10c + d \leq 10e + f\].

We will show that
\[10a + b \leq 10c + f\].

Since
\[10a + b \leq 10c + d\],
\[10a + b \leq 10c + f\]
Thus

$R$

is transitive on

$S$

$.?

6.

$R$

is not an equivalence relation since it is not reflexive, symmetric, and transitive.

Example $\PageIndex{6}$

Let

$A = \mathbb{Z}\setminus\{0\}$

. Define a relation $R$
on $A$

, by

$a R b$

if and only if

$ab > 0$

.?

Solution

1. $R$

is reflexive on $A$

.?

Proof:

Let

$a \in \mathbb{Z}\setminus\{0\}$

we will show that

$a \cdot a > 0$

.?

Clearly
\[ a^2 > 0 \]
since
\[ a \neq 0 \]
and a negative integer multiplied by a negative integer is a positive integer in \( \mathbb{Z} \).

Since
\[ aRa \]
, 
\[ R \]
is reflexive on \( A \).

\textbf{Proof:}

We will show that if 
\[ aRb \]
, then 
\[ bRa \]
.

Let 
\[ a, b \in A \]
s.t.
\[ aRb \]
, that is 
\[ ab > 0 \]
.

Since 
\[ ab = ba \in \mathbb{Z} \]
Since

\[ ba > 0 \]

is symmetric on \( A \).

3.

\( R \)
is not antisymmetric on \( A \).

Counter Example:

Let

\[ a = 3 \]

and

\[ b = 4 \]

then

\[ 3 \cdot 4 > 0 \]

and

\[ 4 \cdot 3 > 0 \]

Since

\[ 3 \neq 4 \]

\( R \)
is not antisymmetric on \( A \).

4.
$R$ is transitive on $A$.

**Proof:**

Let $a, b \in A$ s.t. $ab > 0$ and $b, c \in A$ s.t. $bc > 0$.

There are two cases to be examined:

**Case 1:**

$a > 0$ and $b > 0$.

Since $b > 0$,

c > 0

thus

$ac > 0$.

**Case 2:**

$a < 0$ and $b < 0$.

Since $b < 0$,
\( c < 0 \)

thus

\( ac > 0 \).

Since

\((a \ R \ c)\)
in both possible cases

\( R \)
is transitive on

\( A \).

5. Since

\( R \)
is reflexive, symmetric and transitive, it is an equivalence relation. Equivalence classes are

\([1]\)
and

\([-1]\).

Let

\( a \in A \)
, then

\([a] = \{x \in A : x \sim a\}\)

\( = \{x \in A : xa > 0\}\).

Case 1:

\( a > 0 \)
, then

\( x > 0 \).

\([a] = \{x \in A : x > 0\}\).
\[ [1] = \{1, 2, 3, \ldots \} \]

Case 2:
\[
\alpha < 0
\]
then
\[
x < 0
\]

\[
[a] = \{x \in A : x < 0\}
\]

\[
[-1] = \{-3, -2, -1\}
\]

Note this is a partition since
\[
[x] \cap [y] = \emptyset
\]
or
\[
[x] = [y], \forall x, y \in S
\]
So we have all the intersections are empty.

\[
[1] \cap [-1] = \emptyset
\]

Further, we have
\[
S = \bigcup_{x \in S} [x]
\]
Note that
\[
\emptyset
\]
is excluded from
\[
S
\]

Hasse Diagram

Definition

Let S be a non empty set and let \((R)\) be a partial order relation on \((S)\). Then two elements \((a)\) to \((b)\) of \((S)\) are connected if \((a R b)\). This diagram is called Hasse diagram.