4.3: Divergence of a Series

Skills to Develop

• Explain divergence

In Theorem 3.2.1 we saw that there is a rearrangement of the alternating Harmonic series which diverges to $\infty$ or $-\infty$. In that section we did not fuss over any formal notions of divergence. We assumed instead that you are already familiar with the concept of divergence, probably from taking calculus in the past.

However we are now in the process of building precise, formal definitions for the concepts we will be using so we define the divergence of a sequence as follows.

Definition \(\PageIndex{1}\)

A sequence of real numbers \((s_n)_{n=1}^{\infty}\) diverges if it does not converge to any \(a \in \mathbb{R}\).

It may seem unnecessarily pedantic of us to insist on formally stating such an obvious definition. After all “converge” and “diverge” are opposites in ordinary English. Why wouldn’t they be mathematically opposite too? Why do we have to go to the trouble of formally defining both of them? Since they are opposites defining one implicitly defines the other doesn’t it?

One way to answer that criticism is to state that in mathematics we always work from precisely stated definitions and tightly reasoned logical arguments.

But this is just more pedantry. It is a way of saying, “Because we said so” all dressed up in imposing language. We need to do better than that.
One reason for providing formal definitions of both convergence and divergence is that in mathematics we frequently co-opt words from natural languages like English and imbue them with mathematical meaning that is only tangentially related to the original English definition. When we take two such words which happen to be opposites in English and give them mathematical meanings which are not opposites it can be very confusing, especially at first.

This is what happened with the words “open” and “closed.” These are opposites in English: “not open” is “closed,” “not closed” is “open,” and there is nothing which is both open and closed. But recall that an open interval on the real line, \((a,b)\), is one that does not include either of its endpoints while a closed interval, \([a,b]\), is one that includes both of them.

These may seem like opposites at first but they are not. To see this observe that the interval \((a,b)\) is neither open nor closed since it only contains one of its endpoints. If “open” and “closed” were mathematically opposite then every interval would be either open or closed.

Mathematicians have learned to be extremely careful about this sort of thing. In the case of convergence and divergence of a series, even though these words are actually opposites mathematically (every sequence either converges or diverges and no sequence converges and diverges) it is better to say this explicitly so there can be no confusion.

A sequence \(\{(a_n)_{n=1}\}^{\infty}\) can only converge to a real number, a, in one way: by getting arbitrarily close to a. However there are several ways a sequence might diverge.

Example \(\PageIndex{1}\):

Consider the sequence, \(\{(n)_{n=1}\}^{\infty}\). This clearly diverges by getting larger and larger ... Ooops! Let’s be careful. The sequence \(\left\{(\frac{1}{n})_{n=1}\right\}^{\infty}\) gets larger and larger too, but it converges. What we meant to say was that the terms of the sequence \(\{(n)_{n=1}\}^{\infty}\) become arbitrarily large as \(\{n\}\) increases.

This is clearly a divergent sequence but it may not be clear how to prove this formally. Here’s one way.

To show divergence we must show that the sequence satisfies the negation of the definition of convergence. That is, we must show that for every \(\{r \in \mathbb{R}\}\) there is an \(\{\epsilon > 0\}\) such that for every \(\{N \in \mathbb{N}\}\), there is an \(\{n > N\}\) with \(\{|n-r| \geq \epsilon\}\).

So let \(\{\epsilon = 1\}\), and let \(\{r \in \mathbb{R}\}\) be given. Let \(\{N = r + 2\}\). Then for every \(\{n > N \mid n - r > |(r + 2) - r| = 2 > 1\}\). Therefore the sequence diverges.

This seems to have been rather more work than we should have to do for such a simple problem. Here’s another way which highlights this particular type of divergence.

First we’ll need a new definition:

Definition \(\PageIndex{2}\)

A sequence, \(\{(a_n)_{n=1}\}^{\infty}\), diverges to positive infinity if for every real number \(\{r\}\), there is a real number \(\{N\}\) such that \(\{n > N \mid a_n > r\}\).
A sequence, \(((a_n)_{n=1}^{\infty})\), diverges to negative infinity if for every real number \(r\), there is a real number \(N\) such that \(n > N \Rightarrow a_n < r\).

A sequence is said to diverge to infinity if it diverges to either positive or negative infinity.

In practice we want to think of \(|r|\) as a very large number. This definition says that a sequence diverges to infinity if it becomes arbitrarily large as \(n\) increases, and similarly for divergence to negative infinity.

**Exercise \(\PageIndex{1}\)**

Show that \(((n)_{n=1}^{\infty})\) diverges to infinity.

**Exercise \(\PageIndex{2}\)**

Show that if \(((a_n)_{n=1}^{\infty})\) diverges to infinity then \(((a_n)_{n=1}^{\infty})\) diverges.

We will denote divergence to infinity as

\[
\lim_{n \to \infty} a_n = \pm \infty
\]

However, strictly speaking this is an abuse of notation since the symbol \(\infty\) does not represent a real number. This notation can be very problematic since it looks so much like the notation we use to denote convergence: \(\lim_{n \to \infty} a_n = a\).

Nevertheless, the notation is appropriate because divergence to infinity is “nice” divergence in the sense that it shares many of the properties of convergence, as the next problem shows.

**Exercise \(\PageIndex{3}\)**

Suppose \(\lim_{n \to \infty} a_n = \infty\) and \(\lim_{n \to \infty} b_n = \infty\).

a. Show that \(\lim_{n \to \infty} a_n + b_n = \infty\)

b. Show that \(\lim_{n \to \infty} a_n b_n = \infty\)

c. Is it true that \(\lim_{n \to \infty} \frac{a_n}{b_n} = \infty\)? Explain.

Because divergence to positive or negative infinity shares some of the properties of convergence it is easy to get careless with it. Remember that even though we write \(\lim_{n \to \infty} a_n = \infty\) this is still a divergent sequence in the sense that \(\lim_{n \to \infty} a_n\) does not exist. The symbol \(\infty\) does not represent a real number. This is just a convenient notational shorthand telling us that the sequence diverges by becoming arbitrarily large.

**Exercise \(\PageIndex{4}\)**

Suppose \(\lim_{n \to \infty} a_n = \infty\) and \(\lim_{n \to \infty} b_n = -\infty\) and \(\alpha \in \mathbb{R}\). Prove or give a counterexample:

a. \(\lim_{n \to \infty} a_n + b_n = \infty\)
b. \(\lim_{n \to \infty} a_n b_n = \infty\)

c. \(\lim_{n \to \infty} \alpha a_n = \infty\)

d. \(\lim_{n \to \infty} \alpha a_n = -\infty\)

Finally, a sequence can diverge in other ways as the following problem displays.

Exercise \(\PageIndex{5}\)

Show that each of the following sequences diverge.

a. \(\left((-1)^n \right)_{n=1}^\infty\)

b. \(\left((-1)^n n \right)_{n=1}^\infty\)

c. \(a_n = \begin{cases} 1 & \text{ if } n=2^p \text{ for some } p \in \mathbb{N} \\ \frac{1}{n} & \text{ otherwise } \end{cases}\)

Exercise \(\PageIndex{6}\)

Suppose that \((a_n)_{n=1}^\infty\) diverges but not to infinity and that \(\alpha\) is a real number. What conditions on \(\alpha\) will guarantee that:

a. \((\alpha a_n)_{n=1}^\infty\) converges?

b. \((\alpha a_n)_{n=1}^\infty\) diverges?

Exercise \(\PageIndex{7}\)

Show that if \(|r| > 1\) then \((r^n)_{n=1}^\infty\) diverges. Will it diverge to infinity?

References

1 It is also true that \((-\infty, \infty)\) is both open and closed, but an explanation of this would take us too far a field.

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