7.2: Proof of the Intermediate Value Theorem

Skills to Develop

- Proof of Intermediate value theorem

We now have all of the tools to prove the Intermediate Value Theorem (IVT).

Theorem \(\PageIndex{1}\): Intermediate Value Theorem

Suppose \(f(x)\) is continuous on \([a,b]\) and \(v\) is any real number between \(f(a)\) and \(f(b)\). Then there exists a real number \(c \in [a,b]\) such that \(f(c) = v\).

**Sketch of Proof**

We have two cases to consider: \(f(a) \leq v \leq f(b)\) and \(f(a) \geq v \geq f(b)\). Then there exists a real number \(c \in [a,b]\) such that \(f(c) = v\).

We will look at the case \(f(a) \leq v \leq f(b)\). Let \((x_1 = a)\) and \((y_1 = b)\), so we have \((x_1 \leq y_1)\) and \((f(x_1) \leq v \leq f(y_1))\). Let \((m_1)\) be the midpoint of \((x_1,y_1)\) and notice that we have either \((f(m_1) \leq v)\) or \((f(m_1) \geq v)\). If \((f(m_1) \leq v)\), then we relabel \((x_2 = m_1)\) and \((y_2 = y_1)\). If \((f(m_1) \geq v)\), then we relabel \((x_2 = x_1)\) and \((y_2 = m_1)\). In either case, we end up with \((x_1 \leq x_2 \leq y_2 \leq y_1)\); \(y_2 - x_2 = \frac{1}{2} (y_1 - x_1)\), \((f(x_1) \leq v \leq f(y_1))\), and \((f(x_2) \leq v \leq f(y_2))\).

Now play the same game with the interval \((x_2,y_2)\). If we keep playing this game, we will generate two sequences \((x_n)\) and \((y_n)\), satisfying all of the conditions of the nested interval property. These sequences will also satisfy the following extra property: \((x_n \leq y_n)\); \((f(x_n) \leq v \leq f(y_n))\). By the NIP, there exists a \((c)\) such that \((x_n \leq c \leq y_n)\); \((f(c) = v)\). This should be the \((c)\) that we seek though this is not obvious. Specifically, we need to show that \((f(c) = v)\). This should
be where the continuity of \(f\) at \((c)\) and the extra property on \((x_n)) \text{and } \((y_n))\) come into play.

Exercise \(\PageIndex{1}\)

Turn the ideas of the previous paragraphs into a formal proof of the IVT for the case \(f(a) \leq v \leq f(b)\).

Exercise \(\PageIndex{2}\)

We can modify the proof of the case \(f(a) \leq v \leq f(b)\) into a proof of the IVT for the case \(f(a) \geq v \geq f(b)\). However, there is a sneakier way to prove this case by applying the IVT to the function \(-f\). Do this to prove the IVT for the case \(f(a) \geq v \geq f(b)\).

Exercise \(\PageIndex{3}\)

Use the IVT to prove that any polynomial of odd degree must have a real root.

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