9.1: Trigonometric Series

Skills to Develop

• Explain the Trigonometric series

As we have seen, when they converge, power series are very well behaved and Fourier (trigonometric) series are not necessarily. The fact that trigonometric series were so interesting made them a lightning rod for mathematical study in the late nineteenth century.

For example, consider the question of uniqueness. We saw in Chapter 5 that if a function could be represented by a power series, then that series must be the Taylor series. More precisely, if

\[ f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n, \text{ then } a_n = \frac{f^{(n)}(a)}{n!} \]

But what can be said about the uniqueness of a trigonometric series? If we can represent a function \( f \) as a general trigonometric series

\[ f(x) = \sum_{n=0}^{\infty} (a_n\cos n\pi x + b_n\sin n\pi x) \]

then must this be the Fourier series with the coefficients as determined by Fourier?

For example, if \( \sum_{n=0}^{\infty} (a_n\cos n\pi x + b_n\sin n\pi x) \) converges to \( f(x) \) uniformly on the interval \( (0,1) \), then because of the uniform convergence, Fourier’s term-by-term integration which we saw in earlier is perfectly legitimate and the coefficients are, of necessity, the coefficients he computed. However we have seen that the convergence of a Fourier series need not be uniform. This does not mean that we cannot integrate term-by-term, but it does say that we can’t be sure that term-by-term integration of a Fourier series will yield the integral of the associated function.
This led to a generalization of the integral by Henri Lebesgue in 1905. Lebesgue’s profound work settled the issue of whether or not a bounded pointwise converging trigonometric series is the Fourier series of a function, but we will not go in this direction. We will instead focus on work that Georg Cantor did in the years just prior. Cantor’s work was also profound and had far reaching implications in modern mathematics. It also leads to some very weird conclusions.

To begin, let’s suppress the underlying function and suppose we have

\[\sum_{n=0}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x) = \sum_{n=0}^{\infty} (a'_n \cos n\pi x + b'_n \sin n\pi x)\]

We ask: If these two series are equal must it be true that \(a_n = a'_n\) and \(b_n = b'_n\)? We can reformulate this uniqueness question as follows: Suppose

\[\sum_{n=0}^{\infty} ((a_n - a'_n) \cos n\pi x + (b_n - b'_n) \sin n\pi x) = 0\]

If we let \(c_n = a_n - a'_n\) and \(d_n = b_n = b'_n\), then the question becomes: If \(\sum_{n=0}^{\infty} (c_n \cos n\pi x + d_n \sin n\pi x) = 0\), then will \(c_n = d_n = 0\)? It certainly seems reasonable to suppose so, but at this point we have enough experience with infinite sums to know that we need to be very careful about relying on the intuition we have honed on finite sums.

The answer to this seemingly basic question leads to some very profound results. In particular, answering this question led the mathematician Georg Cantor (1845-1918) to study the makeup of the real number system. This in turn opened the door to the modern view of mathematics of the twentieth century. In particular, Cantor proved the following result in 1871.

Theorem \(\PageIndex{1}\): Cantor

If the trigonometric series

\[\sum_{n=0}^{\infty} (c_n \cos n\pi x + d_n \sin n\pi x) = 0\]

"with the exception of certain values of \((x)\),” then all of its coefficients vanish.

In his attempts to nail down precisely which “certain values” could be exceptional, Cantor was led to examine the nature of subsets of real numbers and ultimately to give a precise definition of the concept of infinite sets and to define an arithmetic of “infinite numbers.” (Actually, he called them transfinite numbers because, by definition, numbers are finite.)

As a first step toward identifying those “certain values,” Cantor proved the following theorem, which we will state but not prove.

Theorem \(\PageIndex{2}\): Cantor, 1870

If the trigonometric series

\[\sum_{n=0}^{\infty} (c_n \cos n\pi x + d_n \sin n\pi x) = 0\]

for all \(x \in \mathbb{R}\) then all of its coefficients vanish.
He then extended this to the following:

**Theorem (PageIndex{3}): Cantor, 1871**

If the trigonometric series

\[
\sum_{n=0}^{\infty} (c_n\cos n\pi x + d_n\sin n\pi x) = 0
\]

for all but finitely many \(x \in \mathbb{R}\) then all of its coefficients vanish.

Observe that this is not a trivial generalization. Although the exceptional points are constrained to be finite in number, this number could still be extraordinarily large. That is, even if the series given above differed from zero on \(10^{10^{100000}}\) distinct points in the interval \((0,10^{10^{-100000}})\) the coefficients still vanish. This remains true even if at each of these \(10^{10^{100000}}\) points the series converges to \(10^{10^{100000}}\). This is truly remarkable when you think of it this way.

![Figure](PageIndex{1}): Georg Cantor

At this point Cantor became more interested in these exceptional points than in the Fourier series problem that he’d started with. The next task he set for himself was to see just how general the set of exceptional points could be. Following Cantor’s lead we make the following definitions.

**Definition (PageIndex{1}):**

Let \(S \subseteq \mathbb{R}\) and let \(a\) be a real number. We say that \(a\) is a limit point (or an accumulation point) of \(S\) if there is a sequence \((a_n)\) with \(a_n \in S - \{a\}\) which converges to \(a\).

**Exercise (PageIndex{1}):**

Let \(S \subseteq \mathbb{R}\) and let \(a\) be a real number. Prove that \(a\) is a limit point of \(S\) if and only if for every \(\varepsilon > 0\) the intersection

\[
[(a-\varepsilon, a+\varepsilon) \cap S - \{a\}] \neq \emptyset
\]

The following definition gets to the heart of the matter.

Definition: Let \( S \subseteq \mathbb{R} \). The set of all limit points of \( \{S\}' \) is called the derived set of \( \{S\} \). The derived set is denoted \( \{S\}' \).

Don’t confuse the derived set of a set with the derivative of a function. They are completely different objects despite the similarity of both the language and notation. The only thing that they have in common is that they were somehow “derived” from something else.

Exercise

Determine the derived set, \( \{S\}' \), of each of the following sets.

a. \( S = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \right\} \)
b. \( S = \left\{ 0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \right\} \)
c. \( S = (0,1] \)
d. \( S = [0,1/2) \cup (1/2,1] \)
e. \( S = \mathbb{Q} \)
f. \( S = \mathbb{R} - \mathbb{Q} \)
g. \( S = \mathbb{Z} \)
h. Any finite set \( \{S\} \).

Exercise

Let \( S \subseteq \mathbb{R} \).

a. Prove that \( \{\{S\}'\}' \subseteq \{S\}' \).
b. Give an example where these two sets are equal.
c. Give an example where these two sets are not equal.

The notion of the derived set forms the foundation of Cantor’s exceptional set of values. Specifically, let \( \{S\} \) again be a set of real numbers and consider the following sequence of sets:

\[
\{S\}' \supseteq \{\{S\}'\}' \supseteq \{\{\{S\}'\}'\}' \supseteq \cdots
\]

Cantor showed that if, at some point, one of these derived sets is empty, then the uniqueness property still holds. Specifically, we have:

Theorem: Cantor, 1871

Let \( \{S\} \) be a subset of the real numbers with the property that one of its derived sets is empty. Then if the trigonometric series \( \sum_{n=0}^{\infty} \{c_n \cos n\pi x + d_n \sin n\pi x\} \) is zero for all \( x \in \mathbb{R} - \{S\} \), then all of the coefficients of the series vanish.
References

1 ‘Weird’ does not mean false. It simply means that some of Cantor’s results can be hard to accept, even after you have seen the proof and verified its validity.

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