11.3: Fourier-Legendre Series

Since Legendre’s equation is *self-adjoint*, we can show that \( (P_n(x)) \) forms an orthogonal set of functions. To decompose functions as series in Legendre polynomials we shall need the integrals

\[
\int_{-1}^{1} P_n^2(x) \, dx = \frac{2n+1}{2},
\]

which can be determined using the relation 5. twice to obtain a recurrence relation

\[
\begin{aligned}
\int_{-1}^{1} & P_n^2(x) \, dx = \int_{-1}^{1} P_n(x) \frac{(2n-1)x P_{n-1}(x)-(n-1)P_{n-2}(x)}{n} \, dx \\
&= \frac{(2n-1)}{n} \int_{-1}^{1} x P_n(x) P_{n-1}(x) \, dx \\
&= \frac{(2n-1)}{n} \int_{-1}^{1} \frac{(n+1)P_{n+1}(x) + n P_{n-1}(x)}{2n+1} P_{n-1}(x) \, dx \\
&= \frac{(2n-1)}{2n+1} \int_{-1}^{1} P_{n-1}^2(x) \, dx,
\end{aligned}
\]

and the use of a very simple integral to fix this number for \( n=0 \),

\[
\int_{-1}^{1} P_0^2(x) \, dx = \frac{2}{2} = 2.
\]

So we can now develop any function on \([-1,1]\) in a *Fourier-Legendre series*

\[
\begin{aligned}
f(x) &= \sum_n A_n P_n(x) \\
A_n &= \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) \, dx
\end{aligned}
\]

Exercise \(\PageIndex{1}\): Fourier-Legendre series

Find the Fourier-Legendre series for
\[ f(x) = \begin{cases} 0, & -1 < x < 0 \lor 1, & 0 < x < 1 \end{cases} \]

**Answer**

We find
\[
\begin{aligned}
A_0 &= \frac{1}{2} \int_0^1 P_0(x) \, dx = \frac{1}{2}, \\
A_1 &= \frac{3}{2} \int_0^1 P_1(x) \, dx = \frac{1}{4}, \\
A_2 &= \frac{5}{2} \int_0^1 P_2(x) \, dx = 0, \\
A_3 &= \frac{7}{2} \int_0^1 P_3(x) \, dx = -\frac{7}{16}.
\end{aligned}
\]

All other coefficients for even \( n \) are zero, for odd \( n \) they can be evaluated explicitly.