11.2: Properties of Legendre Polynomials

Let \( F(x,t) \) be a function of the two variables \( x \) and \( t \) that can be expressed as a Taylor’s series in \( t \), \( \sum_n c_n(x) t^n \). The function \( F \) is then called a generating function of the functions \( c_n(x) \).

Exercise \( \PageIndex{1} \)

Show that \( F(x,t) = \frac{1}{1-xt} \) is a generating function of the polynomials \( x^n \).

Answer

Look at \( \frac{1}{1-xt} = \sum_{n=0}^\infty x^n t^n \) \(|xt|<1\). \( \nonumber \)

Exercise \( \PageIndex{2} \)

Show that \( F(x,t) = e^{\frac{tx-t}{2t}} \) is the generating function for the Bessel functions,

\[ F(x,t) = e^{\frac{tx-t}{2t}} = \sum_{n=0}^\infty J_n(x) t^n. \]

Answer

TBA

Exercise \( \PageIndex{4} \)

(The case of most interest here) \( F(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^\infty P_n(x) t^n. \)
Rodrigues’ Generating Formula

\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n. \]

properties of Legendre Polynomials

1. \((P_\text{even}(n))(x)\) is even or odd if \(n\) is even or odd.
2. \((P_\text{odd}(n))(1)=1\).
3. \((P_\text{odd}(n)(-1))=(-1)^n\).
4. \(((2n+1) P_\text{even}(n))(x) = P_{\text{even}(n+1)}(x)-P_{\text{even}(n-1)}(x))\).
5. \(((2n+1) x P_{\text{odd}}(n)(x) = (n+1) P_{\text{odd}(n+1)}(x) + n P_{\text{odd}(n-1)}(x))\).
6. \(\int_{-1}^{1} P_n(x') dx' = \frac{1}{2n+1} \left[ P_{n+1}(x) - P_{n-1}(x) \right] \).

Let us prove some of these relations, first Rodrigues’ formula (Equation \ref{Rodrigues}). We start from the simple formula

\[(x^{2}-1) \frac{d}{dx} (x^{2}-1)^{n} - 2 n x (x^{2}-1)^{n}=0,\]

which is easily proven by explicit differentiation. This is then differentiated \(n+1\) times,

\[
\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}}\left[ (x^{2}-1) \frac{d}{dx} (x^{2}-1)^{n} - 2 n x (x^{2}-1)^{n}\right]
&= n(n+1) \frac{d^{n}}{dx^{n}}(x^2-1)^n + 2(n+1) x \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n+(x^2-1) \frac{d^{n+2}}{dx^{n+2}} (x^2-1)^n \\
&= -n(n+1) \frac{d^{n}}{dx^{n}}(x^2-1)^n + 2 x \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n+(x^2-1) \frac{d^{n+2}}{dx^{n+2}} (x^2-1)^n \\
&= -\left[\frac{d}{dx}(1-x^2)\frac{d}{dx}\left\{\frac{d^{n}}{dx^{n}}(x^2-1)^n\right\}\right] =0.
\end{aligned}
\]

We have thus proven that \(\frac{d^n}{dx^n}(x^2-1)^n\) satisfies Legendre’s equation. The normalization follows from the evaluation of the highest coefficient,

\[\frac{d^n}{dx^n} x^{2n} = \frac{2n!}{n!} x^{n},\]

and thus we need to multiply the derivative with \(\frac{1}{2^n n!}\) to get the properly normalized \(P_n\).

Let’s use the generating function to prove some of the other properties: 2.\( F(1,t) = \frac{1}{1-t} = \sum_n t^n\) has all coefficients one, so \((P_n(1)=1)\). Similarly for 3.\( F(-1,t) = \frac{1}{1+t} = \sum_n (-1)^n t^n\). Property 5. can be found by differentiating the generating function with respect to \(t\):

\[
\begin{aligned}
\frac{d}{dt} \frac{1}{\sqrt{1-2tx +t^2}} &= \frac{d}{dt} \sum_{n=0}^\infty t^n P_n(x) \\
&= -\frac{1}{2}(1-x^2) \frac{d}{dx} \left\{\frac{d^n}{dx^n}(x^2-1)^n\right\} + n(n+1) \frac{d^n}{dx^n}(x^2-1)^n.
\end{aligned}
\]
\[
\frac{x-t}{(1-2tx+t^2)^{1.5}} \quad \text{and} \quad \sum_{n=0}^\infty n t^{n-1} P_n(x) \\
\sum_{n=0}^\infty t^n P_n(x) - \sum_{n=0}^\infty t^{n+1} P_n(x) = \sum_{n=0}^\infty nt^n P_n(x) - \sum_{n=0}^\infty nt^{n+1} P_n(x) \\
\sum_{n=0}^\infty t^n(2n+1)x P_n(x) = \sum_{n=0}^\infty (n+1)t^n P_{n+1}(x) + \sum_{n=0}^\infty n t^{n-1} P_{n-1}(x).
\]

Equating terms with identical powers of \(t\) we find \((2n+1)x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)\).

Proofs for the other properties can be found using similar methods.