11.2: Properties of Legendre Polynomials

Let \( F(x,t) \) be a function of the two variables \( x \) and \( t \) that can be expressed as a Taylor’s series in \( t \), \( \sum_{n} c_n(x) t^n \). The function \( F \) is then called a generating function of the functions \( \{ c_n \} \).

Exercise \( \PageIndex{1} \)

Show that \( F(x,t) = \frac{1}{1-xt} \) is a generating function of the polynomials \( \{ x^n \} \).

Answer

Look at \( \frac{1}{1-xt} = \sum_{n=0}^\infty x^n t^n \) \((|xt|<1)\).

Exercise \( \PageIndex{2} \)

Show that \( F(x,t) = \exp(\frac{tx-t}{2t}) \) is the generating function for the Bessel functions, \( \{ J_n(x) \} \).

Answer

TBA

Exercise \( \PageIndex{4} \)

(The case of most interest here) \( F(x,t) = \sqrt{1-2xt+t^2} = \sum_{n=0}^\infty P_n(x) t^n \)
Rodrigues’ Generating Formula

\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n. \label{Rodrigues} \]

properties of Legendre Polynomials

1. \((P_n(x))\) is even or odd if \(n\) is even or odd.
2. \((P_n(1)) = 1\).
3. \((P_n(-1)) = (-1)^n\).
4. \(((2n+1)P_n(x)) = P_{n+1}(x)-P_{n-1}(x)\).
5. \(((2n+1)x P_n(x)) = (n+1) P_{n+1}(x) + n P_{n-1}(x)\).
6. \((\int_{-1}^{1} P_n(x') dx') = \frac{1}{2n+1} \left[P_{n+1}(x) - P_{n-1}(x)\right]\).

Let us prove some of these relations, first Rodrigues’ formula (Equation \ref{Rodrigues}). We start from the simple formula

\[(x^2-1) \frac{d}{dx} (x^2-1)^n - 2n x (x^2-1)^n = 0,\]

which is easily proven by explicit differentiation. This is then differentiated \(n+1\) times,

\[
\begin{aligned}
&\frac{d^{n+1}}{dx^{n+1}}\left[(x^2-1) \frac{d}{dx} (x^2-1)^n - 2n x (x^2-1)^n\right] \\
&= n(n+1) \frac{d^n}{dx^n} (x^2-1)^n + 2(n+1) x \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n + (x^2-1) \frac{d^{n+2}}{dx^{n+2}} (x^2-1)^n - 2n(n+1) \frac{d^n}{dx^n} (x^2-1)^n - 2n x \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n \\
&= n(n+1) \frac{d^n}{dx^n} (x^2-1)^n + 2 x \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n + (x^2-1) \frac{d^{n+2}}{dx^{n+2}} (x^2-1)^n - \left[\frac{d}{dx}(1-x^2) \frac{d}{dx}\left(\frac{d^n}{dx^n} (x^2-1)^n\right)\right] = 0.
\end{aligned}
\]

We have thus proven that \((\frac{d^n}{dx^n} (x^2-1)^n)\) satisfies Legendre’s equation. The normalization follows from the evaluation of the highest coefficient,

\[\left(\frac{d^n}{dx^n} x^2\right) x^2 = \frac{2n!}{n!} x^n,\]

and thus we need to multiply the derivative with \((\frac{1}{2^n n!})\) to get the properly normalized \((P_n)\).

Let’s use the generating function to prove some of the other properties: 2. \(\forall(F(1,t) = \frac{1}{1-t} = \sum_n t^n\) has all coefficients one, so \((P_n(1)) = 1\). Similarly for 3. \((F(-1,t) = \frac{1}{1+t} = \sum_n (-1)^n t^n\) Property 5. can be found by differentiating the generating function with respect to \(t\):
\[
\frac{x-t}{(1-2tx+t^2)^{1.5}} = \sum_{n=0}^\infty n t^{n-1} P_n(x)
\]
\[
\frac{x-t}{1-2xt +t^2} = \sum_{n=0}^\infty t^n P_n(x)
\]
\[
\sum_{n=0}^\infty t^n x P_n(x) - \sum_{n=0}^\infty t^{n+1} P_n(x) = \sum_{n=0}^\infty nt^{n-1} P_n(x) - 2 \sum_{n=0}^\infty nt^n x P_n(x) + \sum_{n=0}^\infty (n+1)t^n P_{n+1}(x) + \sum_{n=0}^\infty nt^n P_{n-1}(x)
\]
Equating terms with identical powers of \(t\) we find
\[
(2n+1)x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)
\]
Proofs for the other properties can be found using similar methods.