11.2: Properties of Legendre Polynomials

Let \( F(x,t) \) be a function of the two variables \( x \) and \( t \) that can be expressed as a Taylor’s series in \( t \), \( \sum_{n} c_{n}(x) t^{n} \). The function \( F \) is then called a generating function of the functions \( c_{n}(x) \).

Exercise \( \PageIndex{1} \)

Show that \( F(x,t) = \frac{1}{1-xt} \) is a generating function of the polynomials \( x^{n} \).

**Answer**

Look at \( \frac{1}{1-xt} = \sum_{n=0}^\infty x^{n}t^{n} \; (|xt|<1) \).

Exercise \( \PageIndex{2} \)

Show that \( F(x,t) = \exp\left(\frac{tx-t}{2t}\right) \) is the generating function for the Bessel functions,

\[ F(x,t) = \exp\left(\frac{tx-t}{2t}\right) = \sum_{n=0}^\infty J_{n}(x)t^{n}. \]

**Answer**

TBA

Exercise \( \PageIndex{4} \)

(The case of most interest here) \( F(x,t) = \frac{1}{\sqrt{1-2xt+t^{2}}} = \sum_{n=0}^\infty P_{n}(x) t^{n} \).
Rodrigues’ Generating Formula

\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n. \]

properties of Legendre Polynomials

1. \(P_n(x)\) is even or odd if \(n\) is even or odd.
2. \(P_n(1)=1\).
3. \(P_n(-1)=(-1)^n\).
4. \((2n+1) P_n(x) = P_{n+1}(x)-P_{n-1}(x)\).
5. \((2n+1)x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)\).
6. \(\int_{-1}^{1} P_n(x') dx' = \frac{1}{2n+1} \left[ P_{n+1}(x)-P_{n-1}(x) \right]\).

Let us prove some of these relations, first Rodrigues’ formula (Equation \ref{Rodrigues}). We start from the simple formula

\[(x^2-1) \frac{d}{dx} (x^2-1)^n - 2 n x (x^2-1)^n = 0,\]

which is easily proven by explicit differentiation. This is then differentiated \(n+1\) times,

\[
\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}} \left[ (x^2-1) \frac{d}{dx} (x^2-1)^n - 2 n x (x^2-1)^n \right] &= n(n+1) \frac{d^n}{dx^n} (x^2-1)^n + 2 (n+1) x \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n + (x^2-1) \frac{d^{n+2}}{dx^{n+2}} (x^2-1)^n \\
&= -n(n+1) \frac{d^n}{dx^n} (x^2-1)^n + 2 x \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n + (x^2-1) \frac{d^{n+2}}{dx^{n+2}} (x^2-1)^n \\
&= -\left[ \frac{d}{dx} (1-x^2) \frac{d}{dx} \left\{ \frac{d^n}{dx^n} (x^2-1)^n \right\} \right] + n(n+1) \left\{ \frac{d^n}{dx^n} (x^2-1)^n \right\} = 0.
\end{aligned}
\]

We have thus proven that \(\frac{d^n}{dx^n} x (x^2-1)^n\) satisfies Legendre’s equation. The normalization follows from the evaluation of the highest coefficient,

\[\frac{d^n}{dx^n} x^n = \frac{2^n n!}{n!} x^n,\]

and thus we need to multiply the derivative with \(\frac{1}{2^n n!}\) to get the properly normalized \(P_n\).

Let’s use the generating function to prove some of the other properties: 2.: \(F(1,t) = \frac{1}{1-t} = \sum_n t^n\) has all coefficients one, so \(P_n(1)=1\). Similarly for 3.: \(F(-1,t) = \frac{1}{1+t} = \sum_n (-1)^n t^n\) Property 5. can be found by differentiating the generating function with respect to \(t\):
\[
\frac{x-t}{(1-2tx+t^{2})^{1.5}} = \sum_{n=0}^\infty n t^{n-1} P_n(x)
\]

Equating terms with identical powers of \(t\) we find
\[(2n+1)x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x).\]

Proofs for the other properties can be found using similar methods.