10.5: Properties of Bessel functions

Bessel functions have many interesting properties:

\[
\begin{aligned}
J_{0}(0) &= 1, \\
J_{\nu}(x) &= 0 \quad \text{(if } \nu > 0\text{),} \\
J_{-n}(x) &= (-1)^{n}J_{n}(x), \\
\frac{d}{dx} \left[x^{-\nu}J_{\nu}(x) \right] &= -x^{-\nu}J_{\nu+1}(x), \\
\frac{d}{dx} \left[x^{\nu}J_{\nu}(x) \right] &= x^{\nu}J_{\nu-1}(x), \\
\frac{d}{dx} \left[J_{\nu}(x) \right] &= \frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right], \\
x J_{\nu+1}(x) &= 2 \nu J_{\nu}(x) - x J_{\nu-1}(x), \\
\int x^{-\nu}J_{\nu+1}(x)\,dx &= -x^{-\nu}J_{\nu}(x) + C, \\
\int x^{\nu}J_{\nu-1}(x)\,dx &= x^{\nu}J_{\nu}(x) + C.
\end{aligned}
\]

Let me prove a few of these. First notice from the definition that \(J_{n}(x)\) is even or odd if \(n\) is even or odd,

\[
J_{n}(x) = \sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k}.
\]

Substituting \(x=0\) in the definition of the Bessel function gives \(0\) if \(\nu > 0\), since in that case we have the sum of positive powers of \(0\), which are all equally zero.

Let’s look at \(J_{-n}(x)\):

\[
\begin{aligned}
J_{-n}(x) &= \sum_{k=n}^{\infty}\frac{(-1)^{k+n}}{(k+n)!k!} \left(\frac{x}{2}\right)^{n+2k} \\
&= \sum_{l=0}^{\infty}\frac{(-1)^{l+n}}{(l+n)!l!} \left(\frac{x}{2}\right)^{n+2l} \\
&= (-1)^{n} J_{n}(x).
\end{aligned}
\]

Here we have used the fact that since \(\Gamma(-l) = \pm \infty\), \(1/\Gamma(-l) = 0\) [this can also be proven by defining a recurrence relation for \(1/\Gamma(l)\)]. Furthermore we changed summation variables to \(l=-n+k\).

The next one:
\[
\frac{d}{dx} \left[x^{-\nu} J_{\nu}(x) \right] = 2^{-\nu} \frac{d}{dx} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k! \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k} \right\} \\
= 2^{-\nu} \sum_{k=1}^{\infty} \frac{(-1)^{k}}{(k-1)! \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k-1} \\
= -2^{-\nu} \sum_{l=0}^{\infty} \frac{(-1)^{l}}{(l)! \Gamma(\nu+1+l+1)} \left(\frac{x}{2}\right)^{2l+1} \\
= -2^{-\nu} \sum_{l=0}^{\infty} \frac{(-1)^{l}}{(l)! \Gamma(\nu+1+l+1)} \left(\frac{x}{2}\right)^{2l+\nu+1} \\
= -x^{-\nu} J_{\nu+1}(x).
\]

Similarly
\[
\frac{d}{dx} \left[x^{\nu} J_{\nu}(x) \right] = x^{\nu} J_{\nu-1}(x).
\]

The next relation can be obtained by evaluating the derivatives in the two equations above, and solving for \(J_{\nu}(x)\):
\[
\begin{aligned}
 x^{-\nu} J'_{\nu}(x) - \nu x^{-\nu-1} J_{\nu}(x) &= -x^{-\nu} J_{\nu+1}(x), \\
x^{\nu} J_{\nu}(x) + \nu x^{\nu-1} J_{\nu}(x) &= x^{\nu} J_{\nu-1}(x).
\end{aligned}
\]

Multiply the first equation by \(x^{\nu}\) and the second one by \(x^{-\nu}\) and add:
\[
\begin{aligned}
 -2\nu \frac{1}{x} J_{\nu}(x) &= -J_{\nu+1}(x) + J_{\nu-1}(x) \\
2 J'_{\nu}(x) &= J_{\nu+1}(x) + J_{\nu-1}(x).
\end{aligned}
\]

Integrating the differential relations leads to the integral relations.

Bessel function are an inexhaustible subject – there are always more useful properties than one knows. In mathematical physics one often uses specialist books.