10.5: Properties of Bessel functions

Bessel functions have many interesting properties:

\[
\begin{aligned}
J_{0}(0) &= 1, \\
J_{\nu}(x) &= 0 \quad \text{(if \(\nu > 0\)),} \\
J_{-n}(x) &= (-1)^{n}J_{n}(x), \\
\frac{d}{dx} \left[x^{-\nu}J_{\nu}(x) \right] &= -x^{-\nu}J_{\nu+1}(x), \\
\frac{d}{dx} \left[x^{\nu}J_{\nu}(x) \right] &= x^{\nu}J_{\nu-1}(x), \\
\frac{d}{dx} \left[J_{\nu}(x) \right] &= \frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right], \\
x J_{\nu+1}(x) &= 2\nu J_{\nu}(x) - x J_{\nu-1}(x), \\
\int x^{-\nu}J_{\nu+1}(x)\,dx &= -x^{-\nu}J_{\nu}(x) + C, \\
\int x^{\nu}J_{\nu-1}(x)\,dx &= x^{\nu}J_{\nu}(x) + C.
\end{aligned}
\]

Let me prove a few of these. First notice from the definition that \(J_{n}(x)\) is even or odd if \(n\) is even or odd,

\[
J_{n}(x) = \sum_{k=0}^\infty \frac{(-1)^{k}i^{n}}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k}.
\]

Substituting \(x=0\) in the definition of the Bessel function gives 0 if \(\nu > 0\), since in that case we have the sum of positive powers of 0, which are all equally zero.

Let’s look at \(J_{-n}(x)\):

\[
\begin{aligned}
J_{-n}(x) &= \sum_{k=0}^\infty \frac{(-1)^{k}i^{-n}}{k!(n+k)!} \left(\frac{x}{2}\right)^{-n+2k} \\
&= \sum_{l=0}^\infty \frac{(-1)^{n}}{(l+n)!l!} \left(\frac{x}{2}\right)^{n+2l} \\
&= (-1)^{n}J_{n}(x).
\end{aligned}
\]

Here we have used the fact that since \(\Gamma(-l) = \pm \infty\), \(1/\Gamma(-l) = 0\) [this can also be proven by defining a recurrence relation for \(1/\Gamma(l)\)]. Furthermore we changed summation variables to \(l=-n+k\).

The next one:
\[ \frac{d}{dx} \left[ x^{-\nu} J_{\nu}(x) \right] = 2^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k! \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k} \]

\[ = 2^{-\nu} \sum_{k=1}^{\infty} \frac{(-1)^{k}}{(k-1)! \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k-1} \]

\[ = -2^{-\nu} \sum_{l=0}^{\infty} \frac{(-1)^{l}}{(l!) \Gamma(\nu+l+2)} \left(\frac{x}{2}\right)^{2l+1} \]

\[ = -2^{-\nu} \sum_{l=0}^{\infty} \frac{(-1)^{l}}{(l)! \Gamma(\nu+1+l+1)} \left(\frac{x}{2}\right)^{2l+\nu+1} \]

\[ = -x^{-\nu} J_{\nu+1}(x). \]

Similarly

\[ \frac{d}{dx} \left[ x^\nu J_{\nu}(x) \right] = x^\nu J_{\nu-1}(x). \]

The next relation can be obtained by evaluating the derivatives in the two equations above, and solving for \( J_{\nu}(x) \):

\[ x^{-\nu} J'_{\nu}(x) - \nu x^{-\nu-1} J_{\nu}(x) = -x^{-\nu} J_{\nu+1}(x), \]

\[ x^\nu J_{\nu}(x) + \nu x^\nu J_{\nu}(x) = x^\nu J_{\nu-1}(x). \]

Multiply the first equation by \( x^\nu \) and the second one by \( x^{-\nu} \) and add:

\[ -2\nu x^\nu J_{\nu}(x) = -J_{\nu+1}(x) + J_{\nu-1}(x). \]

After rearrangement of terms this leads to the desired expression.

Eliminating \( J_{\nu}(x) \) between the equations gives (same multiplication, take difference instead): \[ -2\nu x^\nu J_{\nu}(x) = -J_{\nu+1}(x) + J_{\nu-1}(x). \]

Integrating the differential relations leads to the integral relations.

Bessel functions are an inexhaustible subject – there are always more useful properties than one knows. In mathematical physics one often uses specialist books.