11.3: Weighted Voting

We’ll begin with some basic vocabulary for weighted voting systems.

Vocabulary for Weighted Voting

Each individual or entity casting a vote is called a player in the election. They’re often notated as \(\{P_1, P_2, P_3, \ldots, P_N\}\) where \(N\) is the total number of voters.

Each player is given a weight, which usually represents how many votes they get.

The quota is the minimum weight needed for the votes or weight needed for the proposal to be approved.

A weighted voting system will often be represented in a shorthand form:

\(\langle q: w_1, w_2, w_3, \ldots, w_N \rangle\)

In this form, \(\langle q \rangle\) is the quota, \(\langle w_1 \rangle\) is the weight for player 1, and so on.

Example 1

In a small company, there are 4 shareholders. Mr. Smith has a 30% ownership stake in the company, Mr. Garcia has a 25% stake, Mrs. Hughes has a 25% stake, and Mrs. Lee has a 20% stake. They are trying to decide whether to open a new location. The company by-laws state that more than 50% of the ownership has to approve any decision like this. This could be represented by the weighted voting system:
Here we have treated the percentage ownership as votes, so Mr. Smith gets the equivalent of 30 votes, having a 30% ownership stake. Since more than 50% is required to approve the decision, the quota is 51, the smallest whole number over 50.

In order to have a meaningful weighted voting system, it is necessary to put some limits on the quota.

**Limits on Quota**

The quota must be more than \(\frac{1}{2}\) the total number of votes.

The quota can’t be larger than the total number of votes.

Why? Consider the voting system \([q; 3, 2, 1]\)

Here there are 6 total votes. If the quota was set at only 3, then player 1 could vote yes, players 2 and 3 could vote no, and both would reach quota, which doesn’t lead to a decision being made. In order for only one decision to reach quota at a time, the quota must be at least half the total number of votes. If the quota was set to 7, then no group of voters could ever reach quota, and no decision can be made, so it doesn’t make sense for the quota to be larger than the total number of voters.

**Try it Now 1**

In a committee there are four representatives from the management and three representatives from the workers’ union. For a proposal to pass, four of the members must support it, including at least one member of the union. Find a voting system that can represent this situation.

**Answer**

If we represent the players as \(\{M_{1}, M_{2}, M_{3}, M_{4}, U_{1}, U_{2}, U_{3}\}\) then we may be tempted to set up a system like \([4: 1, 1, 1, 1, 1, 1, 1]\). While this system would meet the first requirement that four members must support a proposal for it to pass, this does not satisfy the requirement that at least one member of the union must support it.

Consider the voting system \([10: 11, 3, 2]\). Notice that in this system, player 1 can reach quota without the support of any other player. When this happens, we say that player 1 is a **dictator**.

**Dictator**

A player will be a dictator if their weight is equal to or greater than the quota. The dictator can also block any proposal from passing; the other players cannot reach quota without the dictator.
In the voting system \([8: 6, 3, 2]\), no player is a dictator. However, in this system, the quota can only be reached if player 1 is in support of the proposal; player 2 and 3 cannot reach quota without player 1’s support. In this case, player 1 is said to have **veto power**. Notice that player 1 is not a dictator, since player 1 would still need player 2 or 3’s support to reach quota.

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### Veto Power

A player has veto power if their support is necessary for the quota to be reached. It is possible for more than one player to have veto power, or for no player to have veto power.

With the system \([10: 7, 6, 2]\), player 3 is said to be a **dummy**, meaning they have no influence in the outcome. The only way the quota can be met is with the support of both players 1 and 2 (both of which would have veto power here); the vote of player 3 cannot affect the outcome.

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### Dummy

A player is a dummy if their vote is never essential for a group to reach quota.

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### Example 2

In the voting system \([(16: 7, 6, 3, 2)]\), are any players dictators? Do any have veto power? Are any dummies?

**Solution**

No player can reach quota alone, so there are no dictators.

Without player 1, the rest of the players’ weights add to 14, which doesn’t reach quota, so player 1 has veto power. Likewise, without player 2, the rest of the players’ weights add to 15, which doesn’t reach quota, so player 2 also has veto power.

Since player 1 and 2 can reach quota with either player 3 or player 4’s support, neither player 3 or player 4 have veto power. However they cannot reach quota with player 5’s support alone, so player 5 has no influence on the outcome and is a dummy.

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### Try it Now 2

In the voting system \([(q: 10, 5, 3)]\), which players are dictators, have veto power, and are dummies if the quota is 10? 12? 16?

**Answer**

In the voting system \([(q: 10, 5, 3)]\), if the quota is 10, then player 1 is a dictator since they can reach quota without the support of the other players. This makes the other two players automatically dummies.
If the quota is 12, then player 1 is necessary to reach quota, so has veto power. Since at this point either player 2 or player 3 would allow player 1 to reach quota, neither player is a dummy, so they are regular players (not dictators, no veto power, and not a dummy).

If the quota is 16, then no two players alone can reach quota, so all three players have veto power.

To better define power, we need to introduce the idea of a coalition. A coalition is a group of players voting the same way. In the example above, \((\{P_{1}, P_{2}, P_{4}\})\) would represent the coalition of players 1, 2 and 4. This coalition has a combined weight of \((7+6+3 = 16)\), which meets quota, so this would be a winning coalition.

A player is said to be critical in a coalition if them leaving the coalition would change it from a winning coalition to a losing coalition. In the coalition \((\{P_{1}, P_{2}, P_{4}\})\), every player is critical. In the coalition \((\{P_{3}, P_{4}, P_{5}\})\), no player is critical, since it wasn’t a winning coalition to begin with. In the coalition \((\{P_{1}, P_{2}, P_{3}, P_{4}, P_{5}\})\), only players 1 and 2 are critical; any other player could leave the coalition and it would still meet quota.

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**Coalitions and Critical Players**

A coalition is any group of players voting the same way.

A coalition is a **winning coalition** if the coalition has enough weight to meet quota.

A player is **critical** in a coalition if them leaving the coalition would change it from a winning coalition to a losing coalition.

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**Example 3**

In the Scottish Parliament in 2009 there were 5 political parties: 47 representatives for the Scottish National Party, 46 for the Labour Party, 17 for the Conservative Party, 16 for the Liberal Democrats, and 2 for the Scottish Green Party. Typically all representatives from a party vote as a block, so the parliament can be treated like the weighted voting system:

\([(65: 47, 46, 17, 16, 2)]\)

**Solution**

Consider the coalition \((\{P_{1}, P_{3}, P_{4}\})\). No two players alone could meet the quota, so all three players are critical in this coalition.

In the coalition \((\{P_{1}, P_{3}, P_{4}, P_{5}\})\), any player except \((P_{1})\) could leave the coalition and it would still meet quota, so only \(P_{1}\) is critical in this coalition.

Notice that a player with veto power will be critical in every winning coalition, since removing their support would prevent a proposal from passing.
Likewise, a dummy will never be critical, since their support will never change a losing coalition to a winning one.

### Dictators, Veto, Dummies, and Critical Players

A player is a **dictator** if the single-player coalition containing them is a winning coalition.

A player has **veto power** if they are critical in every winning coalition.

A player is a **dummy** if they are not critical in any winning coalition.

The **Banzhaf power index** was originally created in 1946 by Lionel Penrose, but was reintroduced by John Banzhaf in 1965. The power index is a numerical way of looking at power in a weighted voting situation.

### Calculating Banzhaf Power Index

To calculate the Banzhaf power index:

1. List all winning coalitions
2. In each coalition, identify the players who are critical
3. Count up how many times each player is critical
4. Convert these counts to fractions or decimals by dividing by the total times any player is critical

### Example 4

Find the Banzhaf power index for the voting system $([8: 6, 3, 2])$.

**Solution**

We start by listing all winning coalitions. If you aren’t sure how to do this, you can list all coalitions, then eliminate the non-winning coalitions. No player is a dictator, so we’ll only consider two and three player coalitions.

\[
\begin{align*}
\{P_1, P_2\} & \text{ Total weight: 9. Meets quota.} \\
\{P_1, P_3\} & \text{ Total weight: 8. Meets quota.} \\
\{P_2, P_3\} & \text{ Total weight: 5. Does not meet quota.} \\
\{P_1, P_2, P_3\} & \text{ Total weight: 11. Meets quota.}
\end{align*}
\]

Next we determine which players are critical in each winning coalition. In the winning two-player coalitions, both players are critical since no player can meet quota alone. Underlining the critical players to make it easier to count:
In the three-person coalition, either \(P_2\) or \(P_3\) could leave the coalition and the remaining players could still meet quota, so neither is critical. If \(P_1\) were to leave, the remaining players could not reach quota, so \(P_1\) is critical.

Altogether, \(P_1\) is critical 3 times, \(P_2\) is critical 1 time, and \(P_3\) is critical 1 time.

Converting to percents:

\[
P_1 = \frac{3}{5} = 60\%
\]

\[
P_2 = \frac{1}{5} = 20\%
\]

\[
P_3 = \frac{1}{5} = 20\%
\]

**Example 5**

Consider the voting system \([16: 7, 6, 3, 3, 2]\). Find the Banzhaf power index.

**Solution**

The winning coalitions are listed below, with the critical players underlined.

\[
\{\underline{P}_1, \underline{P}_2, \underline{P}_3\}
\]

\[
\{\underline{P}_1, \underline{P}_2, \underline{P}_4\}
\]

\[
\{\underline{P}_1, \underline{P}_2, P_3, P_4\}
\]

\[
\{\underline{P}_1, \underline{P}_2, \underline{P}_3, P_5\}
\]

\[
\{\underline{P}_1, \underline{P}_2, \underline{P}_4, P_5\}
\]

\[
\{\underline{P}_1, \underline{P}_2, P_3, P_4, P_5\}
\]

Counting up times that each player is critical:

\[
P_1 = 6
\]

\[
P_2 = 6
\]
(P_{3}=2)\\
(P_{4}=2)\\
(P_{5}=0)\\

Total of all: 16

Divide each player’s count by 16 to convert to fractions or percents:

(P_{1}=6 / 16=3 / 8=37.5 \%)\\
(P_{2}=6 / 16=3 / 8=37.5 \%)\\
(P_{3}=2 / 16=1 / 8=12.5 \%)\\
(P_{4}=2 / 16=1 / 8=12.5 \%)\\
(P_{5}=0 / 16=0 \%)

The Banzhaf power index measures a player’s ability to influence the outcome of the vote. Notice that player 5 has a power index of 0, indicating that there is no coalition in which they would be critical power and could influence the outcome. This means player 5 is a dummy, as we noted earlier.

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**Example 6**

Revisiting the Scottish Parliament, with voting system \([65: 47, 46, 17, 16, 2]\)), the winning coalitions are listed, with the critical players underlined.

**Solution**

\[
\begin{array}{ll}
\{\underline{P}_1, \underline{P}_2\} & \{\underline{P}_1, \underline{P}_2, P_3\} & \{\underline{P}_1, \underline{P}_2, P_4\} & \{\underline{P}_1, \underline{P}_3, \underline{P}_4\} & \{\underline{P}_1, P_2, P_3, P_4\} & \{\underline{P}_1, P_2, P_3, P_5\} & \{\underline{P}_1, P_2, P_4, P_5\} & \{\underline{P}_1, P_3, P_4, P_5\} & \{\underline{P}_2, P_3, P_4, P_5\} & \{P_1, P_2, P_3, P_4, P_5\} \\
\end{array}
\]

Counting up times that each player is critical:
Interestingly, even though the Liberal Democrats party has only one less representative than the Conservative Party, and 14 more than the Scottish Green Party, their Banzhaf power index is the same as the Scottish Green Party's. In parliamentary governments, forming coalitions is an essential part of getting results, and a party’s ability to help a coalition reach quota defines its influence.

**Try it Now 3**

Find the Banzhaf power index for the weighted voting system $[36: 20, 17, 16, 3]$).

**Answer**

The voting system tells us that the quota is 36, that Player 1 has 20 votes (or equivalently, has a weight of 20), Player 2 has 17 votes, Player 3 has 16 votes, and Player 4 has 3 votes.

A coalition is any group of one or more players. What we're looking for is winning coalitions - coalitions whose combined votes (weights) add to up to the quota or more. So the coalition $\{\text{P}3, \text{P}4\}$ is not a winning coalition because the combined weight is $16+3=19$, which is below the quota.

So we look at each possible combination of players and identify the winning ones:

\[
\begin{array}{ll}
\text{P}1 & \text{P}2 \\
\text{P}1 & \text{P}3 & \text{P}4 \\
\text{P}1 & \text{P}2 & \text{P}3 & \text{P}4 \\
\text{P}2 & \text{P}3 & \text{P}4 \\
\end{array}
\]

Example 7

Banzhaf used this index to argue that the weighted voting system used in the Nassau County Board of Supervisors in New York was unfair. The county was divided up into 6 districts, each getting voting weight proportional to the population in the district, as shown below. Calculate the power index for each district.
Solution

Translated into a weighted voting system, assuming a simple majority is needed for a proposal to pass:

\[\{58: 31, 31, 28, 21, 2, 2\}\]

Listing the winning coalitions and marking critical players:

\[
\begin{array}{llllll}
\{\underline{\text{H}1}, \underline{\text{H}2}\} & \{\underline{\text{H}1}, \underline{\text{OB}}, \text{NH}\} & \{\underline{\text{H}2}, \underline{\text{OB}}, \text{NH}\} & \{\underline{\text{H}1}, \underline{\text{OB}}, \text{GC}\} & \{\underline{\text{H}2}, \underline{\text{OB}}, \text{GC}\} & \{\underline{\text{H}1}, \underline{\text{H}2}, \text{NH}\} & \{\underline{\text{H}1}, \underline{\text{H}2}, \text{LB}\} & \{\underline{\text{H}1}, \underline{\text{H}2}, \text{GC}\} & \{\underline{\text{H}1}, \underline{\text{H}2}, \text{NH}, \text{LB}\} & \{\text{H}1, \text{H}2, \text{OB}\} & \{\text{H}1, \text{H}2, \text{GC}\} & \{\text{H}1, \text{H}2, \text{OB}, \text{NH}\} & \{\text{H}1, \text{H}2, \text{OB}, \text{GC}\}\end{array}
\]
There are a lot of them! Counting up how many times each player is critical,

\[
\begin{array}{|l|l|l|}
\hline \textbf{ District } & \textbf{ Times critical } & \textbf{ Power index } \\
\hline \text { Hempstead #1 } & 16 & 16 / 48=1 / 3=33 \% \\
\hline \text { Hempstead #2 } & 16 & 16 / 48=1 / 3=33 \% \\
\hline \text { Oyster Bay } & 16 & 16 / 48=1 / 3=33 \% \\
\hline \text { North Hempstead } & 0 & 0 / 48=0 \% \\
\hline \text { Long Beach } & 0 & 0 / 48=0 \% \\
\hline \text { Glen Cove } & 0 & 0 / 48=0 \% \\
\hline
\end{array}
\]

It turns out that the three smaller districts are dummies. Any winning coalition requires two of the larger districts.

The weighted voting system that Americans are most familiar with is the Electoral College system used to elect the President. In the Electoral College, states are given a number of votes equal to the number of their congressional representatives (house + senate). Most states give all their electoral votes to the candidate that wins a majority in their state, turning the Electoral College into a weighted voting system, in which the states are the players.