1.2: The Well Ordering Principle and Mathematical Induction

In this section, we present three basic tools that will often be used in proving properties of the integers. We start with a very important property of integers called the well ordering principle. We then state what is known as the pigeonhole principle, and then we proceed to present an important method called mathematical induction.

Definition: The Well Ordering Principle

A least element exist in any non empty set of positive integers.

This principle can be taken as an axiom on integers and it will be the key to proving many theorems. As a result, we see that any set of positive integers is well ordered while the set of all integers is not well ordered.

Definition: The Pigeonhole Principle

If \( s \) objects are placed in \( k \) boxes for \( s > k \), then at least one box contains more than one object.

Suppose that none of the boxes contains more than one object. Then there are at most \( k \) objects. This leads to a contradiction with the fact that there are \( s \) objects for \( s > k \).

The Principle of Mathematical Induction

We now present a valuable tool for proving results about integers. This tool is the principle of mathematical induction.

The First Principle of Mathematical Induction: If a set of positive integers has the property that, if it contains the integer \( k \), then it also contains \( k+1 \), and if this set contains 1 then it must be the set of all positive integers. More generally, a
property concerning the positive integers that is true for \(n=1\), and that is true for the integer \((n+1)\) whenever it is true for the integer \(n\), must be true for all positive integers.

We use the well ordering principle to prove the first principle of mathematical induction

Let \(\mathcal{S}\) be the set of positive integers containing the integer 1, and the integer \((k+1)\) whenever it contains \((k)\). Assume also that \(\mathcal{S}\) is not the set of all positive integers. As a result, there are some integers that are not contained in \(\mathcal{S}\) and thus those integers must have a least element \((\alpha)\) by the well ordering principle. Notice that \((\alpha \neq 1)\) since \((1 \in S)\). But \((\alpha-1 \in S)\) and thus using the property of \((\mathcal{S})\), \((\alpha \in S)\). Thus \((\mathcal{S})\) must contain all positive integers.

We now present some examples in which we use the principle of induction.

Example \(\PageIndex{1}\)

Use mathematical induction to show that \(\forall n \in \mathbb{N} \sum_{j=1}^nj=\frac{n(n+1)}{2}\).

**Solution**

First note that

\[
\sum_{j=1}^1j=1=\frac{1\cdot 2}{2}
\]

and thus the statement is true for \((n=1)\). For the remaining inductive step, suppose that the formula holds for \((n)\), that is \(\sum_{j=1}^nj=\frac{n(n+1)}{2}\). We show that

\[
\sum_{j=1}^{n+1}j=\frac{(n+1)(n+2)}{2}
\]

to complete the proof by induction. Indeed

\[
\sum_{j=1}^{n+1}j=\sum_{j=1}^nj+(n+1)=\frac{n(n+1)}{2}+(n+1)=\frac{(n+1)(n+2)}{2}
\]

and the result follows.

Example \(\PageIndex{2}\)

Use mathematical induction to prove that \((n!) \leq n^n\) for all positive integers \((n)\).

**Solution**

Note that \((1! = 1) \leq 1^1 = 1\). We now present the inductive step. Suppose that \((n! \leq n^n)\) for some \((n)\), we prove that \(((n+1)! \leq (n+1)^{n+1})\). Note that

\[
(n+1)! = (n+1)n! \leq (n+1)n^n < (n+1)(n+1)^n \leq (n+1)^{n+1}
\]

This completes the proof.

**The Second Principle of Mathematical Induction:** A set of positive integers that has the property that for every integer \((k)\),
if it contains all the integers 1 through \(k\) then it contains \(k+1\) and if it contains 1 then it must be the set of all positive integers. More generally, a property concerning the positive integers that is true for \(n=1\), and that is true for all integers up to \(n+1\) whenever it is true for all integers up to \(n\), must be true for all positive integers.

The second principle of induction is also known as the **principle of strong induction**. Also, the first principle of induction is known as the **principle of weak induction**.

To prove the second principle of induction, we use the first principle of induction.

Let \(\mathcal{T}\) be a set of integers containing 1 and such that for every positive integer \(k\), if it contains \(1, 2, ..., k\), then it contains \(k+1\). Let \(\mathcal{S}\) be the set of all positive integers \(k\) such that all the positive integers less than or equal to \(k\) are in \(\mathcal{T}\). Then 1 is in \(\mathcal{S}\), and we also see that \(k+1\) is in \(\mathcal{S}\). Thus \(\mathcal{S}\) must be the set of all positive integers. Thus \(\mathcal{T}\) must be the set of all positive integers since \(\mathcal{S}\) is a subset of \(\mathcal{T}\).

### Exercises

1. Prove using mathematical induction that \(n < 3^n\) for all positive integers \(n\).
2. Show that \(\sum_{j=1}^nj^2=\frac{n(n+1)(2n+1)}{6}\).
3. Use mathematical induction to prove that \(\sum_{j=1}^n(-1)^{j-1}j^2=(-1)^{n-1}n(n+1)/2\).
4. Use mathematical induction to prove that \(\sum_{j=1}^nj^3=\left[\frac{n(n+1)}{2}\right]^2\) for every positive integer \(n\).
5. Use mathematical induction to prove that \(\sum_{j=1}^n(2j-1)=n^2\)
6. Use mathematical induction to prove that \(2^n<n!\) for \(n\geq 4\).
7. Use mathematical induction to prove that \(n^2<n!\) for \(n\geq 4\).

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