3.5: Theorems of Fermat, Euler, and Wilson

In this section we present three applications of congruences. The first theorem is Wilson’s theorem which states that 
\((p-1)!+1\) is divisible by \(p\), for \(p\) prime. Next, we present Fermat’s theorem, also known as Fermat’s little theorem which states that \(a^p\) and \(a\) have the same remainders when divided by \(p\) where \(p \nmid a\). Finally we present Euler’s theorem which is a generalization of Fermat’s theorem and it states that for any positive integer \(m\) that is relatively prime to an integer \(a\),

\[a^{\phi(m)} \equiv 1 \pmod{m}\]

where \(\phi\) is Euler’s \(\phi\)-function. We start by proving a theorem about the inverse of integers modulo primes.

Theorem

Let \(p\) be a prime. A positive integer \(m\) is its own inverse modulo \(p\) if and only if \(p\) divides \(m+1\) or \(p\) divides \(m-1\).

Suppose that \(m\) is its own inverse. Thus \([m.m \equiv 1 \pmod{p}]\) Hence \(p \mid m^2 - 1\). As a result,

\(p \mid (m-1) \mbox{ or } p \mid (m+1)\)

We get that \([m \equiv 1 \pmod{p}]\) or \([m \equiv -1 \pmod{p}]\).

Conversely, suppose that

\([m \equiv 1 \pmod{p}] \mbox{ or } [m \equiv -1 \pmod{p}]\)

Thus
Wilson’s Theorem

If \( p \) is a prime number, then \( (p-1)!+1 \) divides \((p-1)!+1\).

When \( p = 2 \), the congruence holds. Now let \( p > 2 \). Using Theorem 26, we see that for each \( 1 \leq m \leq p \), there is an inverse \( 1 \leq \bar{m} \leq p \) such that \( m \bar{m} \equiv 1 \pmod{p} \). Thus by Theorem 28, we see that the only two integers that have their own inverses are \( 1 \) and \( p-1 \). Hence after coupling the integers from 2 to \( (p-2) \) each with its inverse, we get \[ 2 \cdot 3 \cdot \ldots \cdot (p-2) \equiv 1 \pmod{p}. \]

Note also that the converse of Wilson’s theorem also holds. The converse tells us whether an integer is prime or not.

If \( m \) is a positive integer with \( (m \mid (p-2)!) \) such that \( (m-1)!+1 \equiv 0 \pmod{m} \) then \( m \) is prime.

We now present Fermat’s Theorem or what is also known as Fermat’s Little Theorem. It states that the remainder of \( a^{p-1} \) when divided by a prime \( p \) that doesn’t divide \( a \) is 1. We then state Euler’s theorem which states that the remainder of \( a^{\phi(m)} \) when divided by a positive integer \( m \) that is relatively prime to \( a \) is 1. We prove Euler’s Theorem only because Fermat’s Theorem is nothing but a special case of Euler’s Theorem. This is due to the fact that for a prime number \( p \), \( \phi(p) = p-1 \).

Euler’s Theorem

If \( m \) is a positive integer and \( a \) is an integer such that \( (a,m) = 1 \), then \( a^{\phi(m)} \equiv 1 \pmod{m} \).

Note that \( 3^4 = 81 \equiv 1 \pmod{5} \). Also, \( 2^6 = 64 \equiv 1 \pmod{9} \).

We now present the proof of Euler’s theorem.

Proof

Let \( \{k_1, k_2, \ldots, k_{\phi(m)}\} \) be a reduced residue system modulo \( m \). By Theorem 25, the set \( \{ak_1, ak_2, \ldots, ak_{\phi(m)}\} \) also forms a reduced residue system modulo \( m \). Thus

\[ \{ak_1, ak_2, \ldots, ak_{\phi(m)}\} \equiv a^{\phi(m)} k_1 k_2 \ldots k_{\phi(m)} \pmod{m}. \]

Now since \( (k_i, m) = 1 \) for all \( 1 \leq i \leq \phi(m) \), we have \( (k_1 k_2 \ldots k_{\phi(m)}, m) = 1 \). Hence by Theorem 22 we can cancel the product of \( k_i \)'s on both sides and we get
\[a^{\phi(m)} \equiv 1 \pmod{m}.\]

An immediate consequence of Euler’s Theorem is:

**Fermat’s Theorem**

If \(p\) is a prime and \(a\) is a positive integer with \(p \nmid a\), then \(a^{p-1} \equiv 1 \pmod{p}\).

We now present a couple of theorems that are direct consequences of Fermat’s theorem. The first states Fermat’s theorem in a different way. It says that the remainder of \(a^p\) when divided by \(p\) is the same as the remainder of \(a\) when divided by \(p\). The other theorem determines the inverse of an integer \(a\) modulo \(p\) where \(p \nmid a\).

If \(p\) is a prime number and \(a\) is a positive integer, then \(a^p \equiv a \pmod{p}\).

If \(p \nmid a\), by Fermat’s theorem we know that \(a^{p-1} \equiv 1 \pmod{p}\). Thus, we get \(a^p \equiv a \pmod{p}\). Now if \(p \mid a\), we have \(a^p \equiv a \equiv 0 \pmod{p}\).

If \(p\) is a prime number and \(a\) is an integer such that \(p \nmid a\), then \(a^{p-2}\) is the inverse of \(a\) modulo \(p\).

If \(p \nmid a\), then Fermat’s theorem says that \(a^{p-1} \equiv 1 \pmod{p}\). Hence \(a^{p-2}a \equiv 1 \pmod{p}\). As a result, \(a^{p-2}\) is the inverse of \(a\) modulo \(p\).

**Exercises**

1. Show that 10!+1 is divisible by 11.
2. What is the remainder when 5!25! is divided by 31?
3. What is the remainder when \(5^{100}\) is divided by 7?
4. Show that if \(p\) is an odd prime, then \(2(p-3)! \equiv -1 \pmod{p}\).
5. Find a reduced residue system modulo \(2^m\), where \(m\) is a positive integer.
6. Show that if \(\{a_1,a_2,\ldots,a_{\phi(m)}\}\) is a reduced residue system modulo \(m\), where \(m\) is a positive integer with \(m \neq 2\), then \(a_1+a_2+\ldots+a_{\phi(m)} \equiv 0 \pmod{m}\).
7. Show that if \(a\) is an integer such that \(a\) is not divisible by 3 or such that \(a\) is divisible by 9, then \(a^7 \equiv a \pmod{63}\).

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