7.2: Chebyshev's Functions

We introduce some number theoretic functions which play important role in the distribution of primes. We also prove analytic results related to those functions. We start by defining the Van-Mangolt function

\[ \Omega(n) = \log p \] if \( n = p^m \) and vanishes otherwise.

We define also the following functions, the last two functions are called Chebyshev’s functions.

1. \( \pi(x) = \sum_{p \leq x} 1 \)
2. \( \theta(x) = \sum_{p \leq x} \log p \)
3. \( \psi(x) = \sum_{n \leq x} \Omega(n) \)

Notice that \[ \psi(x) = \sum_{n \leq x} \Omega(n) = \sum_{m=1, \ p^m \leq x} \sum_p \log p = \sum_{m=1}^{\infty} \sum_{p \leq x^{1/m}} \log p. \]

1. \( \pi(10) = 4 \)
2. \( \theta(10) = \log 2 + \log 3 + \log 5 + \log 7 \)
3. \( \psi(10) = \log 2 + \log 2 + \log 3 + \log 2 + \log 3 + \log 2 + \log 3 \)

It is easy to see that \[ \psi(x) = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \ldots + \theta(x^{1/m}) \] where \( m \leq \log_2 x \). This remark is left as an exercise.

Notice that the above sum will be a finite sum since for some \( m \), we have that \( x^{1/m} < 2 \) and thus \( \theta(x^{1/m}) = 0 \).

We use Abel’s summation formula now to express the two functions \( \pi(x) \) and \( \theta(x) \) in terms of integrals.
For \(x \geq 2\), we have

\[
\theta(x) = \pi(x) \log x - \int_{2}^{x} \frac{\pi(t)}{t} \, dt
\]

and

\[
\pi(x) = \frac{\theta(x)}{\log x} + \int_{2}^{x} \frac{\theta(t)}{t \log^2 t} \, dt.
\]

We define the characteristic function \(\chi(n)\) to be \(1\) if \(n\) is prime and \(0\) otherwise. As a result, we can see from the definition of \(\pi(x)\) and \(\theta(x)\) that they can be represented in terms of the characteristic function \(\chi(n)\). This representation will enable us to apply Abel’s summation formula where \(f(n) = \chi(n)\) for \(\theta(x)\) and where \(f(n) = \chi(n) \log n\) for \(\pi(x)\). So we have,

\[
\pi(x) = \sum_{1 \leq n \leq x} \chi(n) \quad \text{and} \quad \theta(x) = \sum_{1 \leq n \leq x} \chi(n) \log n.
\]

Now let \(g(x) = \log x\) in Theorem 84 with \(y = 1\) and we get the desired result for the integral representation of \(\theta(x)\). Similarly we let \(g(x) = 1/\log x\) with \(y = 3/2\) and we obtain the desired result for \(\pi(x)\) since \(\theta(t) = 0\) for \(t < 2\).

We now prove a theorem that relates the two Chebyshev’s functions \(\theta(x)\) and \(\psi(x)\). The following theorem states that if the limit of one of the two functions \(\theta(x)/x\) or \(\psi(x)/x\) exists then the limit of the other exists as well and the two limits are equal.

For \(x > 0\), we have

\[
0 \leq \frac{\psi(x)}{x} - \frac{\theta(x)}{x} \leq \frac{(\log x)^2}{2\sqrt{x}\log 2}.
\]

From Remark 4, it is easy to see that \(0 \leq \psi(x) - \theta(x) = \theta(x^{1/2}) + \theta(x^{1/3}) + \ldots + \theta(x^{1/m})\) where \(m \leq \log_2 x\). Moreover, we have that \(\theta(x) \leq x \log x\). The result will follow after proving the inequality in Exercise 2.

**Exercises**

1. Show that \(\psi(x) = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \ldots + \theta(x^{1/m})\) where \(m \leq \log_2 x\).
2. Show that \(0 \leq \psi(x) - \theta(x) \leq \log^2 2 \log x\) and thus the result of Theorem 86 follows.
3. Show that the following two relations are equivalent \(\pi(x) = \frac{\log x}{\log 2} + \Omega\left(\frac{\log^2 x}{\log x}\right)\) and \(\theta(x) = x + \Omega\left(\frac{\log^2 x}{\log x}\right)\).

**Contributors and Attributions**

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