8.3: The Riemann Zeta Function

The Riemann zeta function \( \zeta(z) \) is an analytic function that is a very important function in analytic number theory. It is (initially) defined in some domain in the complex plane by the special type of Dirichlet series given by

\[ \zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}, \] where \( \text{Re}(z) > 1 \). It can be readily verified that the given series converges locally uniformly, and thus that \( \zeta(z) \) is indeed analytic in the domain in the complex plane \( \mathbb{C} \) defined by \( \text{Re}(z) > 1 \), and that this function does not have a zero in this domain.

We first prove the following result which is called the Euler Product Formula.

\[ \zeta(z) \text{, as defined by the series above, can be written in the form} \]

\[ \zeta(z) = \prod_{n=1}^\infty \frac{1}{1 - \frac{1}{p_n^z}}, \]

where \( \{p_n\} \) is the sequence of all prime numbers.

Knowing that if \( |x| < 1 \) then

\[ \frac{1}{1-x} = \sum_{k=0}^\infty x^k, \]

one finds that each term \( \frac{1}{1 - \frac{1}{p_n^z}} \) in \( \zeta(z) \) is given by

\[ \frac{1}{1 - \frac{1}{p_n^z}} = \sum_{k=0}^\infty \frac{1}{p_n^{kz}}, \] since every \( |1/p_n^z| < 1 \) if \( \text{Re}(z) > 1 \). This gives that for any integer \( \langle N \rangle \)

\[ \prod_{n=1}^N \frac{1}{1 - \frac{1}{p_n^z}} \]

one finds that each term

\[ \prod_{n=1}^N \frac{1}{1 - \frac{1}{p_n^z}} \]

where \( \text{(i) ranges over} \langle 1, \text{cdots,} N \rangle, \) and \( \text{(j) ranges from} \langle 0 \rangle \) to \( \langle \text{infty} \rangle \), and thus the integers \( \{n\} \) in the third line above range over all integers whose prime number factorization consist of a product of powers of the primes \( \langle p_1 = 2, \text{cdots,} p_N \rangle \). Also note that each such integer \( \{n\} \) appears only once in the sum above.

Now since the series in the definition of \( \zeta(z) \) converges absolutely and the order of the terms in the sum does not matter for the limit, and since, eventually, every integer \( \{n\} \) appears on the right hand side of 8.15 as \( \langle N \rangle \longmapsto \langle \text{infty} \rangle \), then
\[
\lim_{N \to \infty} \left[ \sum_{n^z} \frac{1}{n^z} \right]_{N=\zeta(z)}.
\]
Moreover, \[
\lim_{N \to \infty} \prod_{n=1}^N \frac{1}{\left(1 - \frac{1}{p_n^z}\right)}
\]
exists, and the result follows.

The Riemann zeta function \(\zeta(z)\) as defined through the special Dirichlet series above, can be continued analytically to an analytic function through out the complex plane \(\mathbb{C}\) except to the point \(z=1\), where the continued function has a pole of order 1. Thus the continuation of \(\zeta(z)\) produces a meromorphic function in \(\mathbb{C}\) with a simple pole at 1. The following theorem gives this result.

\(\zeta(z)\), as defined above, can be continued meromorphically in \(\mathbb{C}\), and can be written in the form
\[
\zeta(z) = \frac{1}{z-1} + f(z),
\]
where \(f(z)\) is entire.

Given this continuation of \(\zeta(z)\), and also given the functional equation that is satisfied by this continued function, and which is \[
\zeta(z) = 2^z\pi^{z-1}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta(1-z),
\]
(see a proof in ), where \(\Gamma(z)\) is the complex gamma function, one can deduce that the continued \(\zeta(z)\) has zeros at the points \(z=-2,-4,-6,\cdots\) on the negative real axis. This follows as such: The complex gamma function \(\Gamma(z)\) has poles at the points \(z=-1,-2,-3,\cdots\) on the negative real line, and thus \(\Gamma(1-z)\) must have poles at \(z=2,3,4,\cdots\) on the positive real axis. And since \(\zeta(z)\) is analytic at these points, then it must be that either \(\sin\left(\frac{\pi z}{2}\right)\) or \(\zeta(1-z)\) must have zeros at the points \(z=2,3,4,\cdots\) to cancel out the poles of \(\Gamma(1-z)\), and thus make \(\zeta(z)\) analytic at these points. And since \(\sin\left(\frac{\pi z}{2}\right)\) has zeros at \(z=2,4,6,\cdots\), but not at \(z=3,5,7,\cdots\), then it must be that \(\zeta(1-z)\) has zeros at \(z=3,5,7,\cdots\). This gives that \(\zeta(z)\) has zeros at \(z=-2,-4,-6,\cdots\).

It also follows from the above functional equation, and from the above mentioned fact that \(\zeta(z)\) has no zeros in the domain where \(\Re(z) > 1\), that these zeros at \(z=-2,-4,-6,\cdots\) of \(\zeta(z)\) are the only zeros that have real parts either less than 0, or greater than 1. It was conjectured by Riemann, \textit{The Riemann Hypothesis}, that every other zero of \(\zeta(z)\) in the remaining strip \(0 \leq \Re(z) \leq 1\), all exist on the vertical line \(\Re(z)=1/2\). This hypothesis was checked for zeros in this strip with very large modulus, but remains without a general proof. It is thought that the consequence of the Riemann hypothesis on number theory, provided it turns out to be true, is immense.

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