8.3: The Riemann Zeta Function

The Riemann zeta function $\zeta(z)$ is an analytic function that is a very important function in analytic number theory. It is (initially) defined in some domain in the complex plane by the special type of Dirichlet series given by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},$$

where $\text{Re}(z) > 1$. It can be readily verified that the given series converges locally uniformly, and thus that $\zeta(z)$ is indeed analytic in the domain in the complex plane $(\bf{bf \ C})$ defined by $(\text{Re}(z) > 1)$, and that this function does not have a zero in this domain.

We first prove the following result which is called the Euler Product Formula.

$\zeta(z)$, as defined by the series above, can be written in the form

$$\zeta(z) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^z}\right),$$

where $\{p_n\}$ is the sequence of all prime numbers.

Knowing that if $|x| < 1$ then $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, one finds that each term $\frac{1}{1-\frac{1}{p_n^z}}$ in $\zeta(z)$ is given by

$$\frac{1}{1-\frac{1}{p_n^z}} = \sum_{k=0}^{\infty} \frac{1}{p_n^{kz}},$$

since every $|1/p_n^z| < 1$ if $\text{Re}(z) > 1$. This gives that for any integer $N$ \begin{aligned} \prod_{n=1}^{N} \frac{1}{\left(1-\frac{1}{p_n^z}\right)} &= \prod_{n=1}^{N} \left(1 + \frac{1}{p_n^z} + \frac{1}{p_n^{2z}} + \cdots\right) \\
&= \sum_{\text{integer factors of } n} \frac{1}{n^z} \\
end{aligned}

where $(i)$ ranges over $(1, cdots, N)$, and $(j)$ ranges from $(0)$ to $(\infty)$, and thus the integers $(n)$ in the third line above range over all integers whose prime number factorization consist of a product of powers of the primes $(p_1 = 2, \cdots, p_N)$. Also note that each such integer $(n)$ appears only once in the sum above.

Now since the series in the definition of $\zeta(z)$ converges absolutely and the order of the terms in the sum does not matter for the limit, and since, eventually, every integer $(n)$ appears on the right hand side of 8.15 as $(N\longrightarrow\infty)$, then
The Riemann zeta function \(\zeta(z)\) as defined through the special Dirichlet series above, can be continued analytically to an analytic function throughout the complex plane \(\mathbb{C}\) except to the point \(z=1\), where the continued function has a pole of order 1. Thus the continuation of \(\zeta(z)\) produces a meromorphic function in \(\mathbb{C}\) with a simple pole at 1. The following theorem gives this result.

\(\zeta(z)\), as defined above, can be continued meromorphically in \(\mathbb{C}\), and can be written in the form

\[
\zeta(z) = \frac{1}{z-1} + f(z),
\]

where \(f(z)\) is entire.

Given this continuation of \(\zeta(z)\), and also given the functional equation that is satisfied by this continued function, and which is

\[
\zeta(z) = 2^z\pi^{z-1}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta(1-z),
\]

(see a proof in ), where \(\Gamma\) is the complex gamma function, one can deduce that the continued \(\zeta(z)\) has zeros at the points \(z=-2,-4,-6,\ldots\) on the negative real axis. This follows as such: The complex gamma function \(\Gamma(z)\) has poles at the points \(z=-1,-2,-3,\ldots\) on the negative real line, and thus \(\Gamma(1-z)\) must have poles at \(z=2,3,\ldots\) on the positive real axis. And since \(\zeta(z)\) is analytic at these points, then it must be that either \(\sin\left(\frac{\pi z}{2}\right)\) or \(\zeta(1-z)\) must have zeros at the points \(z=2,3,\ldots\) to cancel out the poles of \(\Gamma(1-z)\), and thus make \(\zeta(z)\) analytic at these points. And since \(\sin\left(\frac{\pi z}{2}\right)\) has zeros at \(z=2,4,\ldots\), but not at \(z=3,5,\ldots\), then it must be that \(\zeta(1-z)\) has zeros at \(z=3,5,\ldots\). This gives that \(\zeta(z)\) has zeros at \(z=-2,-4,-6,\ldots\).

It also follows from the above functional equation, and from the above mentioned fact that \(\zeta(z)\) has no zeros in the domain where \(\Re(z)>1\), that these zeros at \(z=-2,-4,-6,\ldots\) of \(\zeta(z)\) are the only zeros that have real parts either less than 0, or greater than 1. It was conjectured by Riemann, *The Riemann Hypothesis*, that every other zero of \(\zeta(z)\) in the remaining strip \(0\leq \Re(z)\leq 1\), all exist on the vertical line \(\Re(z)=1/2\). This hypothesis was checked for zeros in this strip with very large modulus, but remains without a general proof. It is thought that the consequence of the Riemann hypothesis on number theory, provided it turns out to be true, is immense.

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