14.6: Higher order Derivatives

In single variable calculus we saw that the second derivative is often useful: in appropriate circumstances it measures acceleration; it can be used to identify maximum and minimum points; it tells us something about how sharply curved a graph is. Not surprisingly, second derivatives are also useful in the multi-variable case, but again not surprisingly, things are a bit more complicated.

It's easy to see where some complication is going to come from: with two variables there are four possible second derivatives. To take a "derivative," we must take a partial derivative with respect to \( \frac{\partial}{\partial x} \) or \( \frac{\partial}{\partial y} \), and there are four ways to do it: \( \frac{\partial}{\partial x} \), \( \frac{\partial}{\partial y} \), \( \frac{\partial}{\partial y} \), \( \frac{\partial}{\partial x} \).

Example \( \PageIndex{1} \)

Compute all four second derivatives of \( f(x,y)=x^2y^2 \).

Solution

Using an obvious notation, we get:

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= 2y^2 \\
\frac{\partial^2 f}{\partial x \partial y} &= 4xy \\
\frac{\partial^2 f}{\partial y \partial x} &= 4xy \\
\frac{\partial^2 f}{\partial y^2} &= 2x^2.
\end{align*}
\]

You will have noticed that two of these are the same, the "mixed partials" computed by taking partial derivatives with respect to both variables in the two possible orders. This is not an accident---as long as the function is reasonably nice, this will always be true.

Theorem \( \PageIndex{1} \): Clairaut's Theorem
If the mixed partial derivatives are continuous, they are equal.

Example \(\PageIndex{2}\)

Compute the mixed partials of \( f=xy/(x^2+y^2)\).

Solution

\[
\begin{align*}
  f_x &= \frac{y^3-x^2y}{(x^2+y^2)^2} \\
  f_{xy} &= -\frac{x^4-6x^2y^2+y^4}{(x^2+y^2)^3}
\end{align*}
\]

We leave \(f_{yx}\) as an exercise.

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