3.2: The Improved Euler Method and Related Methods

In Section 3.1, we saw that the global truncation error of Euler’s method is $O(h)$, which would seem to imply that we can achieve arbitrarily accurate results with Euler’s method by simply choosing the step size sufficiently small. However, this is not a good idea, for two reasons. First, after a certain point decreasing the step size will increase roundoff errors to the point where the accuracy will deteriorate rather than improve. The second and more important reason is that in most applications of numerical methods to an initial value problem

\[ y'=f(x,y), \quad y(x_0)=y_0, \]

the expensive part of the computation is the evaluation of $f$. Therefore we want methods that give good results for a given number of such evaluations. This is what motivates us to look for numerical methods better than Euler’s.

To clarify this point, suppose we want to approximate the value of $e$ by applying Euler’s method to the initial value problem

\[ y'=y, \quad y(0)=1 \]

(with solution $y=e^x$) on $[0,1]$, with $h=1/12$, $1/24$, and $1/48$, respectively. Since each step in Euler’s method requires one evaluation of $f$, the number of evaluations of $f$ in each of these attempts is $n=12$, $24$, and $48$, respectively. In each case we accept $y_n$ as an approximation to $e$. The second column of Table \ref{PageIndex1} shows the results. The first column of the table indicates the number of evaluations of $f$ required to obtain the approximation, and the last column contains the value of $e$ rounded to ten significant figures.

In this section we will study the improved Euler method, which requires two evaluations of $f$ at each step. We’ve used this method with $h=1/6$, $1/12$, and $1/24$. The required number of evaluations of $f$ were 12, 24, and 48, as in the three
applications of Euler’s method; however, you can see from the third column of Table \(\PageIndex{1}\) that the approximation to \(e\) obtained by the improved Euler method with only 12 evaluations of \(f\) is better than the approximation obtained by Euler’s method with 48 evaluations.

In Section 3.3, we will study the Runge-Kutta method, which requires four evaluations of \(f\) at each step. We’ve used this method with \(h=1/3\), \(1/6\), and \(1/12\). The required number of evaluations of \(f\) were again 12, 24, and 48, as in the three applications of Euler’s method and the improved Euler method; however, you can see from the fourth column of Table \(\PageIndex{1}\) that the approximation to \(e\) obtained by the Runge-Kutta method with only 12 evaluations of \(f\) is better than the approximation obtained by the improved Euler method with 48 evaluations.

<table>
<thead>
<tr>
<th>(n)</th>
<th>Euler</th>
<th>Improved Euler</th>
<th>Runge-Kutta</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>2.613035290</td>
<td>2.707188994</td>
<td>2.718069764</td>
<td>2.718281828</td>
</tr>
<tr>
<td>24</td>
<td>2.663731258</td>
<td>2.715327371</td>
<td>2.718266612</td>
<td>2.718281828</td>
</tr>
<tr>
<td>48</td>
<td>2.690496599</td>
<td>2.717519565</td>
<td>2.718280809</td>
<td>2.718281828</td>
</tr>
</tbody>
</table>

*Table \(\PageIndex{1}\): Approximations to \(e\) obtained by three numerical methods.*

### The Improved Euler Method

The improved Euler method for solving the initial value problem Equation \ref{eq:3.2.1} is based on approximating the integral curve of Equation \ref{eq:3.2.1} at \((x_i,y(x_i))\) by the line through \((x_i,y(x_i))\) with slope

\[
\frac{m_i = f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))}{2};\nonumber
\]

that is, \(m_i\) is the average of the slopes of the tangents to the integral curve at the endpoints of \([x_i,x_{i+1}]\). The equation of the approximating line is therefore

\[
\frac{\text{\ref{eq:3.2.2} } y = y(x_i) + \frac{f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))}{2}(x-x_i)}{\text{\ref{eq:3.2.2}}}
\]

Setting \(x=x_{i+1} \Rightarrow x_i+h\) in Equation \ref{eq:3.2.2} yields

\[
\frac{\text{\ref{eq:3.2.3} } y_{i+1} = y(x_i) + \frac{h}{2}(f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1})))}{\text{\ref{eq:3.2.3}}}
\]

as an approximation to \(y(x_{i+1})\). As in our derivation of Euler’s method, we replace \(y(x_i)\) (unknown if \(i>0\)) by its approximate value \(y_i\); then Equation \ref{eq:3.2.3} becomes

\[
\frac{\text{\ref{eq:3.2.3} } y_{i+1} = y_i + \frac{h}{2}(f(x_i, y_i) + f(x_{i+1}, y(x_{i+1})))}{\text{\ref{eq:3.2.3}}}
\]

However, this still will not work, because we do not know \(y(x_{i+1})\), which appears on the right. We overcome this by replacing \(y(x_{i+1})\) by \(y_i + hf(x_i, y_i)\), the value that the Euler method would assign to \(y_{i+1}\). Thus, the improved Euler method starts with the known value \(y(x_0) = y_0\) and computes \(y_1\), \(y_2\), \ldots, \(y_n\) successively.
with the formula

\[
\text{label\{eq:3.2.4\} } \quad y_{i+1} = y_i + \frac{h}{2} \left( f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i)) \right). \\
\]

The computation indicated here can be conveniently organized as follows: given \(y_i\), compute

\[
\begin{aligned}
k_{1i} &= f(x_i, y_i),
k_{2i} &= f(x_i + h, y_i + hk_{1i}),
y_{i+1} &= y_i + \frac{h}{2} (k_{1i} + k_{2i}).
\end{aligned}
\]

The improved Euler method requires two evaluations of \(f(x,y)\) per step, while Euler’s method requires only one. However, we will see at the end of this section that if \(f\) satisfies appropriate assumptions, the local truncation error with the improved Euler method is \(O(h^3)\), rather than \(O(h^2)\) as with Euler’s method. Therefore the global truncation error with the improved Euler method is \(O(h^2)\); however, we will not prove this.

We note that the magnitude of the local truncation error in the improved Euler method and other methods discussed in this section is determined by the third derivative \(y'''\) of the solution of the initial value problem. Therefore the local truncation error will be larger where \(|y'''|\) is large, or smaller where \(|y'''|\) is small.

The next example, which deals with the initial value problem considered in Example \(\PageIndex{1}\), illustrates the computational procedure indicated in the improved Euler method.

**Example \(\PageIndex{1}\)**

Use the improved Euler method with \(h=0.1\) to find approximate values of the solution of the initial value problem

\[
\text{label\{eq:3.2.5\} } \quad y' + 2y = x^3e^{-2x}, \quad y(0) = 1
\]

at \(x=0.1, 0.2, 0.3\).

**Solution**

As in Example 3.1.1, we rewrite Equation \ref{eq:3.2.5} as

\[
\text{\begin{aligned}
y' &= -2y + x^3e^{-2x}, \\
y(0) &= 1,
\end{aligned}\}
\]

which is of the form Equation \ref{eq:3.2.1}, with

\[
\begin{aligned}
f(x,y) &= -2y + x^3e^{-2x}, \\
x_0 &= 0, \quad y_0 = 1.
\end{aligned}
\]

The improved Euler method yields

\[
\begin{aligned}
k_{10} &= f(x_0, y_0) = f(0,1) = -2, \\
k_{20} &= f(x_1, y_0 + hk_{10}) = f(0.1, 1 + 0.1(-2)) = -2.1599181269, \\
y_{1} &= y_0 + \frac{h}{2} (k_{10} + k_{20}), \\
k_{11} &= f(x_1, y_1) = f(0.1, 1 - 0.820040937), \\
k_{21} &= -2.0820040937 + 0.153e^{-0.2} = 1.639263142.
\end{aligned}
\]
\[f(x_2, y_1 + hk_{11}) = f(0.2, 0.820040937 + 0.1(-1.639263142)) = f(0.2, 0.656114622) = -2(0.656114622) + (0.2)^3e^{-0.4} = -1.306866684,\]
\[y_2 = y_1 + \frac{h}{2}(k_{11} + k_{21}),\]
\[= 0.820040937 + (0.05)(-1.639263142 - 1.306866684) = 0.672734445.\]
\[k_{12} = f(x_2, y_2) = f(0.2, 0.672734445) = -2(0.672734445) + (0.2)^3e^{0.4} = -1.340106330,\]
\[k_{22} = f(x_3, y_2 + hk_{12}) = f(0.3, 0.672734445 + 0.1(-1.340106330)) = f(0.3, 0.538723812) = -2(0.538723812) + (0.3)^3e^{-0.6} = -1.062629710,\]
\[y_3 = y_2 + \frac{h}{2}(k_{12} + k_{22}),\]
\[= 0.672734445 + (0.05)(-1.340106330 - 1.062629710) = 0.552597643.\]

Example \(\PageIndex{2}\)

Table \(\PageIndex{2}\) shows results of using the improved Euler method with step sizes \(h=0.1\) and \(h=0.05\) to find approximate values of the solution of the initial value problem

\[y' + 2y = x^3e^{-2x}, \quad y(0) = 1\]

at \(x=0\), \(x=0.1\), \(x=0.2\), \(x=0.3\), ..., \(x=1.0\). For comparison, it also shows the corresponding approximate values obtained with Euler’s method in \(\text{example:3.1.2}\), and the values of the exact solution

\[y = e^{-2x}(x^4 + 4).\]

The results obtained by the improved Euler method with \(h=0.1\) are better than those obtained by Euler’s method with \(h=0.05\).
Table

Table \(\PageIndex{2}\): Numerical solution of \(y'+2y=x^3e^{-2x}, y(0)=1\), by Euler's method and the improved Euler method.

Example \(\PageIndex{3}\)

Table \(\PageIndex{3}\) shows analogous results for the nonlinear initial value problem \[y'=-2y^2+xy+x^2, y(0)=1.\]

We applied Euler’s method to this problem in Example \(\PageIndex{3}\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(h=0.1)</th>
<th>(h=0.05)</th>
<th>(h=0.1)</th>
<th>(h=0.05)</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.000000000</td>
<td>1.000000000</td>
<td>1.000000000</td>
<td>1.000000000</td>
<td>1.000000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.800000000</td>
<td>0.821375000</td>
<td>0.840500000</td>
<td>0.838288371</td>
<td>0.837584494</td>
</tr>
<tr>
<td>0.2</td>
<td>0.681000000</td>
<td>0.707795377</td>
<td>0.733430846</td>
<td>0.730556677</td>
<td>0.729641890</td>
</tr>
<tr>
<td>0.3</td>
<td>0.605867800</td>
<td>0.633776590</td>
<td>0.661600806</td>
<td>0.658552190</td>
<td>0.657580377</td>
</tr>
<tr>
<td>0.4</td>
<td>0.559628676</td>
<td>0.587454526</td>
<td>0.615961841</td>
<td>0.612884493</td>
<td>0.611901791</td>
</tr>
<tr>
<td>0.5</td>
<td>0.535376972</td>
<td>0.562906169</td>
<td>0.591634742</td>
<td>0.588558952</td>
<td>0.587575491</td>
</tr>
<tr>
<td>0.6</td>
<td>0.529820120</td>
<td>0.557143535</td>
<td>0.586006935</td>
<td>0.582927224</td>
<td>0.581942225</td>
</tr>
<tr>
<td>0.7</td>
<td>0.541467455</td>
<td>0.568716935</td>
<td>0.597712120</td>
<td>0.594618012</td>
<td>0.593629526</td>
</tr>
</tbody>
</table>
Example \(\PageIndex{4}\)

Use step sizes \((h=0.2), (h=0.1),\) and \((h=0.05)\) to find approximate values of the solution of \(\tag{3.2.6} y'-2xy=1, \quad y(0)=3\) at \(x=0, 0.2, 0.4, 0.6, \ldots, 2.0\) by (a) the improved Euler method; (b) the improved Euler semilinear method. (We used Euler’s method and the Euler semilinear method on this problem in [example:3.1.4].)

**Solution**

a. Rewriting Equation \ref{eq:3.2.6} as

\[y'=1+2xy, \quad y(0)=3\]

and applying the improved Euler method with \(f(x,y)=1+2xy\) yields the results shown in Table \(\PageIndex{4}\).

b. Since \(y_1=e^{x^2}\) is a solution of the complementary equation \(y'-2xy=0\), we can apply the improved Euler semilinear method to Equation \ref{eq:3.2.6}, with

\[y=ue^{x^2} \quad \text{and} \quad u'=e^{-x^2}, \quad u(0)=3.\]

The results listed in Table \(\PageIndex{5}\) are clearly better than those obtained by the improved Euler method.

\[
\begin{array}{cccccc}
(x) & (h=0.1) & (h=0.05) & (h=0.1) & (h=0.05) & \text{Exact} \\
0.0 & 0.569732776 & 0.596951988 & 0.626008824 & 0.622898279 & 0.621907458 \\
0.2 & 0.614392311 & 0.641457729 & 0.670351225 & 0.667237617 & 0.666250842 \\
0.4 & 0.675192037 & 0.701764495 & 0.730069610 & 0.726985837 & 0.726015790 \\
0.6 & 0.8 & 0.9 & 1.0 & & \\
\end{array}
\]

\[
\textbf{Table \(\PageIndex{3}\): Numerical solution of } y'=-2y^2+xy+x^2, \quad y(0)=1, \text{ by Euler’s method and the improved Euler method.}
\]
A Family of Methods with O(h^3) Local Truncation Error

We will now derive a class of methods with O(h^3) local truncation error for solving Equation \ref{eq:3.2.1}. For
simplicity, we assume that \( f \), \( f_x \), \( f_y \), \( f_{xx} \), \( f_{yy} \), and \( f_{xy} \) are continuous and bounded for all \((x,y)\). This implies that if \( y \) is the solution of Equation \ref{eq:3.2.1} then \( y'' \) and \( y''' \) are bounded \((Exercise 3.2.31)\).

We begin by approximating the integral curve of Equation \ref{eq:3.2.1} at \((x_i,y(x_i))\) by the line through \((x_i,y(x_i))\) with slope

\[ m_i = \sigma y'(x_i) + \rho y'(x_i + \theta h), \]

where \( \sigma \), \( \rho \), and \( \theta \) are constants that we will soon specify; however, we insist at the outset that \( 0 < \theta \leq 1 \), so that

\[ x_i < x_i + \theta h \leq x_{i+1}. \]

The equation of the approximating line is

\[
\begin{array}{rcl}
y & = & y(x_i) + m_i(x - x_i) \\
& = & y(x_i) + \left[ \sigma y'(x_i) + \rho y'(x_i + \theta h) \right](x - x_i).
\end{array}
\]

Setting \( x = x_{i+1} = x_i + h \) in Equation \ref{eq:3.2.7} yields

\[ \hat{y}_{i+1} = y(x_i) + h \left[ \sigma y'(x_i) + \rho y'(x_i + \theta h) \right] \]

as an approximation to \( y(x_{i+1}) \).

To determine \( \sigma \), \( \rho \), and \( \theta \) so that the error

\[
\begin{array}{rcl}
E_i & = & y(x_{i+1}) - \hat{y}_{i+1} \\
& = & y(x_{i+1}) - y(x_i) - h \left[ \sigma y'(x_i) + \rho y'(x_i + \theta h) \right]
\end{array}
\]

in this approximation is \( O(h^3) \), we begin by recalling from Taylor’s theorem that

\[ y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(\hat{x}_i), \]

where \( \hat{x}_i \) is in \((x_i,x_{i+1})\). Since \( y''' \) is bounded this implies that

\[ y(x_{i+1}) - y(x_i) - hy'(x_i) - \frac{h^2}{2}y''(x_i) = O(h^3). \]

Comparing this with Equation \ref{eq:3.2.8} shows that \( (E_i = O(h^3)) \) if

\[
\| y'(x_i + \theta h) - y'(x_i) \| = \frac{h}{2} \overline{y''(x_i)} + O(h^2).
\]

However, applying Taylor’s theorem to \( y'' \) shows that

\[ y(x_i + \theta h) = y(x_i) + \theta h y''(x_i) + \frac{(\theta h)^2}{2} y'''(\overline{x}_i), \]

where \( \overline{x}_i \) is in \((x_i,x_i + \theta h)\).
where \( \overline{x_i} \) is in \((x_i, x_i + \theta h)\). Since \( y'' \) is bounded, this implies that
\[
[y'(x_i + \theta h) = y'(x_i) + \theta h y''(x_i) + O(h^2)].
\]

Substituting this into Equation \ref{eq:3.2.9} and noting that the sum of two \( O(h^2) \) terms is again \( O(h^2) \) shows that
\[
(E_i = O(h^3)) \text{ if}
\]
\[
[(\sigma + \rho)y'(x_i) + \rho \theta h y''(x_i) = y'(x_i) + \frac{h}{2} y''(x_i),]
\]
which is true if
\[
(\sigma + \rho = 1 \quad \text{and} \quad \rho \theta = \frac{1}{2}).
\]

Since \( y = f(x, y) \), we can now conclude from Equation \ref{eq:3.2.8} that
\[
y(x_{i+1}) = y(x_i) + h \left[ \sigma f(x_i, y(x_i)) + \rho f(x_i + \theta h, y(x_i + \theta h)) \right] + O(h^3)
\]
if \( (\sigma), (\rho), \text{ and } (\theta) \) satisfy Equation \ref{eq:3.2.10}. However, this formula would not be useful even if we knew \( y(x_i) \) exactly (as we would for \( i = 0 \)), since we still wouldn't know \( y(x_i + \theta h) \) exactly. To overcome this difficulty, we again use Taylor's theorem to write
\[
y(x_i + \theta h) = y(x_i) + \theta h y'(x_i) + \frac{h^2}{2} y''(\tilde{x_i}),
\]
where \( \tilde{x_i} \) is in \((x_i, x_i + \theta h)\). Since \( y'(x_i) = f(x_i, y(x_i)) \) and \( y'' \) is bounded, this implies that
\[
y(x_i + \theta h) - y(x_i) - \theta h f(x_i, y(x_i)) = O(h^2)
\]
for some constant \( K \). Since \( f_y \) is bounded, the mean value theorem implies that
\[
|f(x_i + \theta h, u) - f(x_i + \theta h, v)| = M |u - v|
\]
for some constant \( M \). Letting
\[
[u = y(x_i + \theta h) \quad \text{and} \quad v = y(x_i) + \theta h f(x_i, y(x_i)) \]
\]
and recalling Equation \ref{eq:3.2.12} shows that
\[
|f(x_i + \theta h, y(x_i + \theta h))| = f(x_i + \theta h, y(x_i)) + O(h^2).
\]

Substituting this into Equation \ref{eq:3.2.11} yields
\[
\begin{aligned}
y(x_{i+1}) & = y(x_i) + h \left[ \sigma f(x_i, y(x_i)) + \rho f(x_i + \theta h, y(x_i + \theta h)) \right] + O(h^3).
\end{aligned}
\]
This implies that the formula
\[ y_{i+1} = y_i + h \left[ \sigma f(x_i, y_i) + \rho f(x_i + \theta h, y_i + \theta hf(x_i, y_i)) \right] \]

has \( O(h^3) \) local truncation error if \( \sigma \), \( \rho \), and \( \theta \) satisfy Equation \ref{eq:3.2.10}. Substituting \( \sigma = 1 - \rho \) and \( \theta = 1/2 \rho \) here yields

\[ y_{i+1} = y_i + h \left[ (1 - \rho)f(x_i, y_i) + \rho f(x_i + \frac{h}{2\rho}, y_i + \frac{h}{2\rho}f(x_i, y_i)) \right]. \]

The computation indicated here can be conveniently organized as follows: given \( y_i \), compute

\[
\begin{aligned}
k_{1i} &= f(x_i, y_i), \\
k_{2i} &= f(x_i + \frac{h}{2\rho}, y_i + \frac{h}{2\rho}k_{1i}), \\
y_{i+1} &= y_i + h \left[ (1 - \rho)k_{1i} + \rho k_{2i} \right].
\end{aligned}
\]

Consistent with our requirement that \( 0 < \theta < 1 \), we require that \( \rho \geq 1/2 \). Letting \( \rho = 1/2 \) in Equation \ref{eq:3.2.13} yields the improved Euler method Equation \ref{eq:3.2.4}. Letting \( \rho = 3/4 \) yields Heun’s method,

\[ y_{i+1} = y_i + h \left[ \frac{1}{4}f(x_i, y_i) + \frac{3}{4}f(x_i + \frac{2h}{3}, y_i + \frac{2h}{3}f(x_i, y_i)) \right], \]

which can be organized as

\[
\begin{aligned}
k_{1i} &= f(x_i, y_i), \\
k_{2i} &= f(x_i + \frac{2h}{3}, y_i + \frac{2h}{3}k_{1i}), \\
y_{i+1} &= y_i + \frac{h}{4}(k_{1i} + 3k_{2i}).
\end{aligned}
\]

Letting \( \rho = 1 \) yields the midpoint method,

\[ y_{i+1} = y_i + hf(x_i + \frac{h}{2}, y_i + \frac{h}{2}f(x_i, y_i)), \]

which can be organized as

\[
\begin{aligned}
k_{1i} &= f(x_i, y_i), \\
k_{2i} &= f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{1i}), \\
y_{i+1} &= y_i + hk_{2i}.
\end{aligned}
\]

Examples involving the midpoint method and Heun’s method are given in Exercises 3.2.23 - 3.3.30.