8.3: Solution of Initial Value Problems

Laplace Transforms of Derivatives

In the rest of this chapter we’ll use the Laplace transform to solve initial value problems for constant coefficient second order equations. To do this, we must know how the Laplace transform of \( f'(t) \) is related to the Laplace transform of \( f(t) \). The next theorem answers this question.

Theorem \( \PageIndex{1} \)

Suppose \( f \) is continuous on \([0,\infty)\) and of exponential order \( s_0 \), and \( f' \) is piecewise continuous on \([0,\infty)\). Then \( f \) and \( f' \) have Laplace transforms for \( s > s_0 \), and

\[
\label{eq:8.3.1} \mathcal{L}(f') = s \mathcal{L}(f) - f(0). \]

Proof

We know from Theorem 8.1.6 that \( \mathcal{L}(f) \) is defined for \( s > s_0 \). We first consider the case where \( f' \) is continuous on \([0,\infty)\). Integration by parts yields

\[
\begin{align}
\int_0^T e^{-st}f'(t)\,dt &= e^{-st}f(t)\big|_0^T + s \int_0^T e^{-st}f(t)\,dt \\
&= e^{-sT}f(T) - f(0) + s \int_0^T e^{-st}f(t)\,dt
\end{align}
\]

for any \( T > 0 \). Since \( f \) is of exponential order \( s_0 \), \( \lim_{T \to \infty} e^{-sT}f(T) = 0 \) and the integral in on the right side of Equation \ref{eq:8.3.2} converges as \( T \to \infty \) if \( s > s_0 \). Therefore
\[
\int_0^\infty e^{-st}f'(t)\,dt = -f(0) + s\int_0^\infty e^{-st}f(t)\,dt,
\]
which proves Equation \ref{eq:8.3.1}.

Suppose \((T>0)\) and \((f'(t))\) is only piecewise continuous on \([0,T]\), with discontinuities at \((t_1 < t_2 < \cdots < t_{n-1})\). For convenience, let \((t_0=0)\) and \((t_n=T)\). Integrating by parts yields

\[
\int_{t_{i-1}}^{t_i} e^{-st}f'(t)\,dt = e^{-st_i}f(t_i) - e^{-st_{i-1}}f(t_{i-1}) + s\int_{t_{i-1}}^{t_i} e^{-st}f(t)\,dt.
\]

Summing both sides of this equation from \((i=1)\) to \((n)\) and noting that

\[
\left(e^{-st_1}f(t_1) - e^{-st_0}f(t_0)\right) + \left(e^{-st_2}f(t_2) - e^{-st_1}f(t_1)\right) + \cdots + \left(e^{-st_N}f(t_N) - e^{-st_{N-1}}f(t_{N-1})\right) = e^{-sT}f(T) - f(0)
\]
yields Equation \ref{eq:8.3.2}, so Equation \ref{eq:8.3.1} follows as before.

Example \(\PageIndex{1}\):

In Example 8.1.4 we saw that

\[
\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}.
\]

Solution

Applying Equation \ref{eq:8.3.1} with \((f(t) = \cos \omega t)\) shows that

\[
\mathcal{L}(-\omega \sin \omega t) = s\mathcal{L}(\cos \omega t) - 1 = -\frac{\omega^2}{s^2 + \omega^2}.
\]

Therefore

\[
\mathcal{L}(\sin \omega t) = -\frac{\omega}{s^2 + \omega^2},
\]
which agrees with the corresponding result obtained in 8.1.4.

In Section 2.1 we showed that the solution of the initial value problem

\[
\begin{align*}
y' &= ay, \quad y(0) = y_0,
\end{align*}
\]
is \((y = y_0 e^{at})\). We’ll now obtain this result by using the Laplace transform.

Let \((Y(s) = \mathcal{L}(y))\) be the Laplace transform of the unknown solution of Equation \ref{eq:8.3.3}. Taking Laplace
transforms of both sides of Equation \ref{eq:8.3.3} yields
\[\{\mathcal{L}(y')=a\{\mathcal{L}(y)\}\] which, by Theorem \(\PageIndex{1}\), can be rewritten as
\[s\{\mathcal{L}(y)-y(0)=a\{\mathcal{L}(y)\}\] or
\[sY(s)-y(0)=aY(s)\] Solving for \(Y(s)\) yields
\[Y(s)={y_0\over s-a}\] so
\[y={\mathcal{L}}^{-1}(Y(s))={\mathcal{L}}^{-1}\left({y_0\over s-a}\right)={y_0}{\mathcal{L}}^{-1}\left({1\over s-a}\right)=y_0e^{at}\] which agrees with the known result.

We need the next theorem to solve second order differential equations using the Laplace transform.

Theorem \(\PageIndex{2}\)

Suppose \(f\) and \(f'\) are continuous on \([0,\infty)\) and of exponential order \(s_0\), and that \(f''\) is piecewise continuous on \([0,\infty)\). Then \(f\), \(f'\), and \(f''\) have Laplace transforms for \(s>s_0\),
\[\label{eq:8.3.4} \mathcal{L}(f')=s\mathcal{L}(f)-f(0)\] and
\[\label{eq:8.3.5} \mathcal{L}(f'')=s^2\mathcal{L}(f)-f'(0)-sf(0)\]

Proof
Theorem \(\PageIndex{1}\) implies that \(\{\mathcal{L}(f')\}\) exists and satisfies Equation \ref{eq:8.3.4} for \((s>s_0)\). To prove that \(\{\mathcal{L}(f'')\}\) exists and satisfies Equation \ref{eq:8.3.5} for \((s>s_0)\), we first apply Theorem \(\PageIndex{1}\) to \(g=f'\). Since \(g'\) satisfies the hypotheses of Theorem \(\PageIndex{1}\), we conclude that \(\{\mathcal{L}(g')\}\) is defined and satisfies
\[\{\mathcal{L}(g')=s\{\mathcal{L}(g)-g(0)\}\] for \((s>s_0)\). However, since \(g''=f''\), this can be rewritten as
Solving Second Order Equations with the Laplace Transform

We’ll now use the Laplace transform to solve initial value problems for second order equations.

Example \( \PageIndex{2} \)

Use the Laplace transform to solve the initial value problem

\[
\label{eq:8.3.6} y''-6y'+5y=3e^{2t}, \quad y(0)=2, \quad y'(0)=3. \]

Solution

Taking Laplace transforms of both sides of the differential equation in Equation \ref{eq:8.3.6} yields

\[
\left[ \{\cal L\}(y'')-6\{\cal L\}(y')+5\{\cal L\}(y) = \frac{3}{s-2} \right], \nonumber
\]

which we rewrite as

\[
\left[ \{\cal L\}(y') = s\{\cal L\}(y)-2 \right], \nonumber
\]

and

\[
\{\cal L\}(y'') = s^2\{\cal L\}(y) - 3 - 2s. \nonumber
\]

Substituting from the last two equations into Equation \ref{eq:8.3.7} yields

\[
\left[ (s^2-6s+5)\{\cal L\}(y) = \frac{3}{s-2} + (3+2s) + 6(-2) \right]. \nonumber
\]

Therefore

\[
\label{eq:8.3.8} (s^2-6s+5)Y(s) = \frac{3}{s-2} + (3+2s) + 6(-2). \]

so

\[
(s-5)(s-1)Y(s) = \left[ 3+(s-2)(2s-9) \right], \nonumber
\]

and
\[ Y(s) = \frac{3 + (s-2)(2s-9)}{(s-2)(s-5)(s-1)} \]

Heaviside’s method yields the partial fraction expansion

\[ Y(s) = -\frac{1}{s-2} + \frac{1}{2} \frac{1}{s-5} + \frac{5}{2} \frac{1}{s-1}, \]

and taking the inverse transform of this yields

\[ y = -e^{2t} + \frac{1}{2} e^{5t} + \frac{5}{2} e^{t} \]

as the solution of Equation \ref{eq:8.3.6}.

It isn’t necessary to write all the steps that we used to obtain Equation \ref{eq:8.3.8}. To see how to avoid this, let’s apply the method of Example \( \PageIndex{2} \) to the general initial value problem

\[ ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \] \tag{\ref{eq:8.3.9}}

Taking Laplace transforms of both sides of the differential equation in Equation \ref{eq:8.3.9} yields

\[ a \mathcal{L}(y'') + b \mathcal{L}(y') + c \mathcal{L}(y) = F(s). \] \tag{\ref{eq:8.3.10}}

Now let \( Y(s) = \mathcal{L}(y) \). Theorem \( \PageIndex{2} \) and the initial conditions in Equation \ref{eq:8.3.9} imply that

\[ \mathcal{L}(y') = sY(s) - k_0 \quad \text{and} \quad \mathcal{L}(y'') = s^2Y(s) - k_1 - k_0s. \]

Substituting these into Equation \ref{eq:8.3.10} yields

\[ a(s^2Y(s) - k_1 - k_0s) + b(sY(s) - k_0) + cY(s) = F(s). \] \tag{\ref{eq:8.3.11}}

The coefficient of \( Y(s) \) on the left is the characteristic polynomial

\[ p(s) = as^2 + bs + c \]

of the complementary equation for Equation \ref{eq:8.3.9}. Using this and moving the terms involving \( (k_0) \) and \( (k_1) \) to the right side of Equation \ref{eq:8.3.11} yields

\[ p(s)Y(s) = F(s) + a(k_1 + k_0s) + bk_0. \] \tag{\ref{eq:8.3.12}}

This equation corresponds to Equation \ref{eq:8.3.8} of Example \( \PageIndex{2} \). Having established the form of this equation in the general case, it is preferable to go directly from the initial value problem to this equation. You may find it easier to remember Equation \ref{eq:8.3.12} rewritten as

\[ p(s)Y(s) = F(s) + a\left(y'(0) + sy(0)\right) + by(0). \]

Example \( \PageIndex{3} \)
Use the Laplace transform to solve the initial value problem

\[
2y''+3y'+y=8e^{-2t}, \quad y(0)=-4; \ y'(0)=2.
\]

**Solution**

The characteristic polynomial is

\[p(s)=2s^2+3s+1=(2s+1)(s+1)\]

and

\[F(s)={\cal L}(8e^{-2t})={8\over s+2},\]

so Equation \ref{eq:8.3.13} becomes

\[(2s+1)(s+1)Y(s)={8\over s+2}+2(2-4s)+3(-4)\]

Solving for \(Y(s)\) yields

\[Y(s)={4\over3}{1-(s+2)(s+1)\over(s+1/2)(s+1)(s+2)}\]

Heaviside's method yields the partial fraction expansion

\[Y(s)={4\over3}{1\over s+1/2}-{8\over s+1}+{8\over3}{1\over s+2}\]

so the solution of Equation \ref{eq:8.3.14} is

\[y={\cal L}^{-1}(Y(s))={4\over3}e^{-t/2}-8e^{-t}+{8\over3}e^{-2t}\]

(Figure \(\PageIndex{1}\)).

\[\text{Figure } \PageIndex{1}: y={4\over3}e^{-t/2}-8e^{-t}+{8\over3}e^{-2t}\]
Solve the initial value problem

\[ y''+2y'+2y=1, \quad y(0)=-3, \quad y'(0)=1. \]

\[ \text{Solution} \]

The characteristic polynomial is

\[ p(s)=s^2+2s+2=(s+1)^2+1 \]

and

\[ F(s)={\cal L}(1)={1\over s}, \]

so Equation \ref{eq:8.3.13} becomes

\[ (s+1)^2+1 \quad Y(s)={1\over s}+1\cdot(1-3s)+2(-3) \]

Solving for \( Y(s) \) yields

\[ Y(s)={1-s(5+3s)\over s((s+1)^2+1)} \]

In Example 8.2.8 we found the inverse transform of this function to be

\[ y={1\over2}-{7\over2}e^{-t}\cos t-{5\over2}e^{-t}\sin t \]

(Figure \( \PageIndex{2} \)), which is therefore the solution of Equation \ref{eq:8.3.15}.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{\( \PageIndex{2} \): \( y={1\over2}-{7\over2}e^{-t}\cos t-{5\over2}e^{-t}\sin t \)}
\end{figure}

Note

In our examples we applied Theorems \( \PageIndex{1} \) and \( \PageIndex{2} \) without verifying that the unknown function \( y \) satisfies their hypotheses. This is characteristic of the formal manipulative way in which the Laplace transform is used.
to solve differential equations. Any doubts about the validity of the method for solving a given equation can be resolved by verifying that the resulting function \( y \) is the solution of the given problem.