8.4: The Unit Step Function

In the next section we’ll consider initial value problems

\[ ay''+by'+cy=f(t), \quad y(0)=k_0, \quad y'(0)=k_1, \quad \text{nonumber} \]

where \(a\), \(b\), and \(c\) are constants and \(f(t)\) is piecewise continuous. In this section we’ll develop procedures for using the table of Laplace transforms to find Laplace transforms of piecewise continuous functions, and to find the piecewise continuous inverses of Laplace transforms.

Example \(\PageIndex{1}\)

Use the table of Laplace transforms to find the Laplace transform of

\[ f(t)=\left\{ \begin{array}{cl} 2t+1, & 0 \le t < 2, \\
& t \ge 2 
\end{array} \right. \]

(Figure \(\PageIndex{1}\)).
Figure \(\PageIndex{1}\): The piecewise continuous function Equation \ref{eq:8.4.1}.

Solution

Since the formula for \(f(t)\) changes at \(t=2\), we write

\[
\begin{array}{ll}
\mathcal{L}(f) = &\int_0^\infty e^{-st} f(t) \, dt \\
= &\int_0^2 e^{-st} (2t+1) \, dt + \int_2^\infty e^{-st} (3t-2t-1) \, dt.
\end{array}
\]

To relate the first term to a Laplace transform, we add and subtract

\[
\int_2^\infty e^{-st} (2t+1) \, dt \nonumber
\]

in Equation \ref{eq:8.4.2} to obtain

\[
\begin{aligned}
\mathcal{L}(f) &= \int_0^\infty e^{-st} (2t+1) \, dt + \int_2^\infty e^{-st} (t-1) \, dt \\
&= \int_0^\infty e^{-st} (2t+1) \, dt + \frac{1}{e^{-2s}} \int_0^\infty e^{-sx} (x+1) \, dx.
\end{aligned}
\]

Since the symbol used for the variable of integration has no effect on the value of a definite integral, we can now replace \(x\) by the more standard \(t\) and write

\[
\begin{aligned}
\int_2^\infty e^{-st} (t-1) \, dt &= e^{-2s} \int_0^\infty e^{-st} (t+1) \, dt.
\end{aligned}
\]

This and Equation \ref{eq:8.4.3} imply that

\[
\begin{align*}
\mathcal{L}(f) &= \mathcal{L}(2t+1) + e^{-2s} \mathcal{L}(t+1) \\
&= \mathcal{L}(2t+1) + e^{-2s} \mathcal{L}(t+1).
\end{align*}
\]
Now we can use the table of Laplace transforms to find that

\[\mathcal{L}(f) = \frac{2}{s^2} + \frac{1}{s} + e^{-2s} \left( \frac{1}{s^2} + \frac{1}{s} \right)\]

### Laplace Transforms of Piecewise Continuous Functions

We’ll now develop the method of Example \(\PageIndex{1}\) into a systematic way to find the Laplace transform of a piecewise continuous function. It is convenient to introduce the **unit step function**, defined as

\[
\begin{align*}
u(t) &= \begin{cases} 
0, & t < 0 \\
1, & t \geq 0.
\end{cases}
\end{align*}
\]

Thus, \(u(t)\) “steps” from the constant value \(0\) to the constant value \(1\) at \(t=0^+\). If we replace \(t\) by \(t-\tau\) in Equation \ref{eq:8.4.4}, then

\[
u(t-\tau) = \begin{cases} 
0, & t < \tau, \\
1, & t \geq \tau
\end{cases};
\]

that is, the step now occurs at \(t=\tau\) (Figure \(\PageIndex{2}\)).

![Figure \(\PageIndex{2}\): \(y=u(t-\tau)\)](image)

The step function enables us to represent piecewise continuous functions conveniently. For example, consider the function

\[
f(t) = \begin{cases} 
f_0(t), & 0 \leq t < t_1, \\
f_1(t), & t \geq t_1
\end{cases}
\]

where we assume that \(f_0\) and \(f_1\) are defined on \([0, \infty)\), even though they equal \(f\) only on the indicated intervals. This assumption enables us to rewrite Equation \ref{eq:8.4.5} as

\[
f(t) = f_0(t) + u(t-t_1)(f_1(t)-f_0(t))
\]

To verify this, note that if \(t < t_1\) then \(u(t-t_1) = 0\) and Equation \ref{eq:8.4.6} becomes

\[
f(t) = f_0(t) + (0)(f_1(t)-f_0(t)) = f_0(t)
\]
If \((t \geq t_1)\) then \(u(t-t_1)=1\) and Equation \[\text{ref:8.4.6}\] becomes

\[
f(t)=f_0(t)+(1)(f_1(t)-f_0(t))=f_1(t).
\]

We need the next theorem to show how Equation \[\text{ref:8.4.6}\] can be used to find \({\cal L}(f)\).

**Theorem \(\PageIndex{1}\)**

Let \(g\) be defined on \([0, \infty)\). Suppose \((\{\tau \geq 0\})\) and \((\{\{\cal L\}\{\left(g(t+\tau)\right)\})\) exists for \((s>s_0)\). Then \((\{\{\cal L\}\{\left(\left(u(t-\tau)g(t)\right)\right)\})\) exists for \((s>s_0)\), and

\[
\{\{\cal L\}\{\left(u(t-\tau)g(t)\right)\}=e^{-s\tau}\{\{\cal L\}\{\left(g(t+\tau)\right)\}.
\]

**Proof**

By definition,

\[
\{\{\cal L\}\{\left(u(t-\tau)g(t)\right)\}={\int_0^\infty e^{-st} \left(u(t-\tau)g(t)\right) dt.
\]

From this and the definition of \((\{\{\cal L\}\{\left(u(t-\tau)\right)\})\),

\[
\{\{\cal L\}\{\left(u(t-\tau)g(t)\right)\}={\int_0^\tau e^{-st}(0) dt + \int_{\tau}^\infty e^{-st}g(t) dt.
\]

The first integral on the right equals zero. Introducing the new variable of integration \((x=t-\tau)\) in the second integral yields

\[
\{\{\cal L\}\{\left(u(t-\tau)g(t)\right)\}={\int_0^\infty e^{-s(x+\tau)}g(x+\tau) dx = e^{-s\tau}{\int_0^\infty e^{-sx}g(x+\tau) dx}.
\]

Changing the name of the variable of integration in the last integral from \((x)\) to \((t)\) yields

\[
\{\{\cal L\}\{\left(u(t-\tau)g(t)\right)\}={\int_0^\infty e^{-s\tau}{\int_0^\infty e^{-st}g(t+\tau) dt} = e^{-s\tau}{\cal L}(g(t+\tau)).
\]

**Example \(\PageIndex{2}\)**

Find \((\{\{\cal L\}\{\left(u(t-1)(t^2+1)\right)\})\).

**Solution**

Here \((\tau=1)\) and \((g(t)=t^2+1)\), so

\[
\{\{\cal L\}\{\left(g(t+1)^2+1\right)\}=t^2+2t+2.
\]

Since

\[
\{\{\cal L\}\{\left(g(t+1)\right)\}={2\over s^3}+{2\over s^2}+{2\over s}.
\]
Theorem \(\PageIndex{1}\) implies that
\[
[\mathcal{L}(u(t-1)(t^2+1)) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s}\right)].
\]

Example \(\PageIndex{3}\)

Use Theorem \(\PageIndex{1}\) to find the Laplace transform of the function
\[
f(t) = \begin{cases} 2t+1, & 0 \leq t < 2, \\ 3t, & t \geq 2, \end{cases}
\]
from Example \(\PageIndex{1}\).

Solution

We first write \(f\) in the form Equation \ref{eq:8.4.6} as
\[
f(t) = 2t+1 + u(t-2)(t-1).
\]
Therefore
\[
\begin{aligned}
\mathcal{L}(f) &= \mathcal{L}(2t+1) + \mathcal{L}(u(t-2)(t-1)) \\
&= \mathcal{L}(2t+1) + e^{-2s}\mathcal{L}(t+1) \\
&= \frac{2}{s^2} + \frac{1}{s} + e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right),
\end{aligned}
\]
which is the result obtained in Example \(\PageIndex{1}\).

Formula Equation \ref{eq:8.4.6} can be extended to more general piecewise continuous functions. For example, we can write
\[
f(t) = f_0(t) + u(t-t_1)(f_1(t) - f_0(t)) + u(t-t_2)(f_2(t) - f_1(t))
\]
as
\[
f(t) = f_0(t) + u(t-t_1) + u(t-t_2)
\]
if \((f_0), (f_1), \text{ and } (f_2)\) are all defined on \(([0, \infty))\).

Example \(\PageIndex{4}\)

Find the Laplace transform of
\[
f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ -2t+1, & 2 \leq t < 3, \\ 3t, & 3 \leq t < 5, \\ t-1, & t \geq 5, \end{cases}
\]
(Figure \(\PageIndex{3}\)).
Figure \(\text{Figure 
PageIndex(3)}\): The piecewise continuous function Equation \ref{eq:8.4.7}

Solution

In terms of step functions,

\[
\begin{aligned}
f(t) &= 1 + u(t-2)(-2t+1-1) + u(t-3)(3t+2t-1) &= +u(t-5)(t-1-3t), \\
\end{aligned}
\]

or

\[
f(t) = 1 - 2u(t-2)t + u(t-3)(5t-1) - u(t-5)(2t+1).
\]

Now Theorem \(\PageIndex{1}\) implies that

\[
\begin{aligned}
\mathcal{L}(f) &= \mathcal{L}(1) - 2e^{-2s}\mathcal{L}(t+2) + e^{-3s}\mathcal{L}(5(t+3)-1) - e^{-5s}\mathcal{L}(2(t+5)+1) \\
&= \frac{1}{s} - 2e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + e^{-3s}\left(\frac{5}{s^2} + \frac{14}{s}\right) - e^{-5s}\left(\frac{2}{s^2} + \frac{11}{s}\right).
\end{aligned}
\]

The trigonometric identities

\[
\begin{align}
\sin(A+B) &= \sin A\cos B + \cos A\sin B \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B
\end{align}
\]

are useful in problems that involve shifting the arguments of trigonometric functions. We’ll use these identities in the next example.

Example \(\PageIndex{5}\)

Find the Laplace transform of

\[
f(t) = \begin{cases} 
\sin t, & 0 \leq t < \frac{\pi}{2} \\
\cos t - 3\sin t, & \frac{\pi}{2} \leq t < \pi 
\end{cases}
\]
In terms of step functions,
\[ f(t) = \sin t + u(t-\pi/2) \left( \cos t - 4 \sin t \right) + u(t-\pi) \left( 2 \cos t + 3 \sin t \right). \]

Now Theorem \ref{thm:8.4.10} implies that
\[
\begin{align*}
\mathcal{L}(f) &= \mathcal{L}(\sin t) + e^{-\pi/2 s} \mathcal{L} \left( \cos \left( t + \frac{\pi}{2} \right) - 4 \sin \left( t + \frac{\pi}{2} \right) \right) \\
&\quad + e^{-\pi s} \mathcal{L} \left( 2 \cos (t+\pi) + 3 \sin (t+\pi) \right).
\end{align*}
\]

Since
\[
\cos \left( t + \frac{\pi}{2} \right) - 4 \sin \left( t + \frac{\pi}{2} \right) = - \sin t - 4 \cos t
\]
and
\[ 2 \cos (t+\pi) + 3 \sin (t+\pi) = -2 \cos t - 3 \sin t, \]
we see from Equation \ref{eq:8.4.11} that
\[
\begin{align*}
\mathcal{L}(f) &= \mathcal{L}(\sin t) - e^{-\pi s/2} \mathcal{L}(\sin t + 4 \cos t) \\
&\quad - e^{-\pi s} \mathcal{L}(2 \cos t + 3 \sin t)
\end{align*}
\]
The Second Shifting Theorem

Replacing \( g(t) \) by \( g(t-\tau) \) in Theorem \( \PageIndex{1} \) yields the next theorem.

**Theorem \( \PageIndex{2} \): Second Shifting Theorem**

If \( \tau \geq 0 \) and \( \mathcal{L}(g) \) exists for \( s > s_0 \) then \( \mathcal{L}(u(t-\tau)g(t-\tau)) \) exists for \( s > s_0 \) and

\[
\mathcal{L}(u(t-\tau)g(t-\tau)) = e^{-s\tau} \mathcal{L}(g(t)),
\]

or, equivalently,

\[
\text{if } g(t) \leftrightarrow G(s), \text{ then } u(t-\tau)g(t-\tau) \leftrightarrow e^{-s\tau}G(s).
\]

Note

Recall that the First Shifting Theorem (Theorem 8.1.3) states that multiplying a function by \( e^{at} \) corresponds to shifting the argument of its transform by \( a \) units. Theorem \( \PageIndex{2} \) states that multiplying a Laplace transform by the exponential \( e^{-\tau s} \) corresponds to shifting the argument of the inverse transform by \( \tau \) units.

**Example \( \PageIndex{6} \)**

Use Equation \ref{eq:8.4.12} to find

\[
\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right).
\]

**Solution**

To apply Equation \ref{eq:8.4.12} we let \( \tau=2 \) and \( G(s)=1/s^2 \). Then \( g(t)=t \) and Equation \ref{eq:8.4.12} implies that

\[
\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right) = u(t-2)(t-2).
\]

**Example \( \PageIndex{7} \)**

Find the inverse Laplace transform \( h(t) \) of

\[
H(s)=\frac{1}{s^2}-e^{-s}\left(\frac{1}{s^2}+\frac{2}{s}\right)+e^{-4s}\left(\frac{4}{s^3}+\frac{1}{s}\right),
\]

and find distinct formulas for \( h(t) \) on appropriate intervals.

**Solution**

Let

\[
G_0(s)=\frac{1}{s^2}, \quad G_1(s)=\frac{1}{s^2}+\frac{2}{s}, \quad G_2(s)=\frac{4}{s^3}+\frac{1}{s}.
\]
Then
\[ g_0(t) = t, \quad g_1(t) = t + 2, \quad g_2(t) = 2t^2 + 1. \]

Hence, Equation \ref{eq:8.4.12} and the linearity of \( \mathcal{L}^{-1} \) imply that
\[
\begin{aligned}
  h(t) &= \mathcal{L}^{-1}(G_0(s)) - \mathcal{L}^{-1}(e^{-s}G_1(s)) + \mathcal{L}^{-1}(e^{-4s}G_2(s)) \\
  &= t - u(t-1)(t+1) + u(t-4)(2t^2 - 16t + 33),
\end{aligned}
\]

which can also be written as
\[
\begin{cases}
  h(t) &= t, &0 \leq t < 1, \\
  &= -1, &1 \leq t < 4, \\
  &= 2t^2 - 16t + 32, &t \geq 4.
\end{cases}
\]

Example \( \PageIndex{8} \)

Find the inverse transform of
\[
H(s) = \frac{2s}{s^2+4} - e^{-\frac{\pi}{2}s} \frac{3s+1}{s^2+9} + e^{-\pi s} \frac{s+1}{s^2+6s+10}.
\]

Solution

Let
\[
G_0(s) = \frac{2s}{s^2+4}, \quad G_1(s) = -\frac{3s+1}{s^2+9},
\]

and
\[
G_2(s) = \frac{s+1}{s^2+6s+10} = \frac{(s+3)-2}{(s+3)^2+1}.
\]

Then
\[
g_0(t) = 2\cos 2t, \quad g_1(t) = -3\cos 3t - \frac{1}{3}\sin 3t,
\]

and
\[
g_2(t) = e^{-3t}. \cos t - 2\sin t.
\]

Therefore Equation \ref{eq:8.4.12} and the linearity of \( \mathcal{L}^{-1} \) imply that
\[
\begin{aligned}
  h(t) &= 2\cos 2t + u(t-\pi/2)\left[3\cos 3(t-\pi/2) + 1\right] + 3\sin 3(t-\pi/2) \sin (t-\pi) \right] \right] \\
  &= -u(t-\pi)e^{-3(t-\pi/2)} \cos 3t \right] \right].
\end{aligned}
\]

Using the trigonometric identities Equation \ref{eq:8.4.8} and Equation \ref{eq:8.4.9}, we can rewrite this as
\[
\begin{aligned}
  h(t) &= 2\cos 2t + u(t-\pi/2)\left[3\sin 3t - 1\right] + u(t-\pi)e^{-3(t-\pi/2)} \cos 3t \right] \right].
\end{aligned}
\]
\[
\pi) \left( \cos t - 2\sin t \right) \end{align}
\]

(Figure \(\PageIndex{5}\)).

Figure \(\PageIndex{5}\): The piecewise continuous function Equation \ref{eq:8.4.13}. 

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