9.1: Introduction to Linear Higher Order Equations

An \(n\)th order differential equation is said to be linear if it can be written in the form

\[ y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x). \]  

We considered equations of this form with \(n=1\) in Section 2.1 and with \(n=2\) in Chapter 5. In this chapter \(n\) is an arbitrary positive integer. In this section we sketch the general theory of linear \(n\)th order equations. Since this theory has already been discussed for \(n=2\) in Sections 5.1 and 5.3, we’ll omit proofs.

For convenience, we consider linear differential equations written as

\[ P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = F(x), \]

which can be rewritten as Equation \ref{eq:9.1.1} on any interval on which \(P_0\) has no zeros, with \(p_1 = P_1/P_0\), …, \(p_n = P_n/P_0\) and \(f = F/P_0\). For simplicity, throughout this chapter we’ll abbreviate the left side of Equation \ref{eq:9.1.2} by \(Ly\); that is,

\[ Ly = P_0y^{(n)} + P_1y^{(n-1)} + \cdots + P_ny. \]

We say that the equation \(Ly = F\) is normal on \((a,b)\) if \((P_0), (P_1), \ldots, (P_n)\) and \(F\) are continuous on \((a,b)\) and \((P_0)\) has no zeros on \((a,b)\). If this is so then \(Ly = F\) can be written as Equation \ref{eq:9.1.1} with \(p_1, \ldots, p_n\) and \(f\) continuous on \((a,b)\).

The next theorem is analogous to Theorem 5.3.1.

Theorem \(\PageIndex{1}\)}
Suppose \( \langle Ly=F \rangle \) is normal on \( \langle (a,b) \rangle \), let \( \langle x_0 \rangle \) be a point in \( \langle (a,b) \rangle \) and let \( \langle k_0 \rangle, \langle k_1 \rangle, \ldots, \langle k_{\{n-1\}} \rangle \) be arbitrary real numbers. Then the initial value problem
\[
\langle Ly=F, \quad y(x_0)=k_0, \quad y'(x_0)=k_1, \ldots, \quad y^{(n-1)}(x_0)=k_{n-1} \rangle
\]
has a unique solution on \( \langle (a,b) \rangle \).

### Homogeneous Equations

Equation \( \text{ref}[eq:9.1.2] \) is said to be **homogeneous** if \( \langle F \equiv 0 \rangle \) and **nonhomogeneous** otherwise. Since \( \langle y \equiv 0 \rangle \) is obviously a solution of \( \langle Ly=0 \rangle \), we call it the **trivial** solution. Any other solution is **nontrivial**.

If \( \langle y_1 \rangle, \langle y_2 \rangle, \ldots, \langle y_n \rangle \) are defined on \( \langle (a,b) \rangle \) and \( \langle c_1 \rangle, \langle c_2 \rangle, \ldots, \langle c_n \rangle \) are constants, then

\[
\langle y_1=c_1y_1+c_2y_2+\cdots+c_ny_n \rangle
\]

is a **linear combination** of \( \langle \{y_1,y_2,\ldots,y_n\} \rangle \). It’s easy to show that if \( \langle y_1 \rangle, \langle y_2 \rangle, \ldots, \langle y_n \rangle \) are solutions of \( \langle Ly=0 \rangle \) on \( \langle (a,b) \rangle \), then so is any linear combination of \( \langle \{y_1,y_2,\ldots,y_n\} \rangle \). (See the proof of Theorem 5.1.2.) We say that \( \langle \{y_1,y_2,\ldots,y_n\} \rangle \) is a **fundamental set of solutions** of \( \langle Ly=0 \rangle \) on \( \langle (a,b) \rangle \) if every solution of \( \langle Ly=0 \rangle \) on \( \langle (a,b) \rangle \) can be written as a linear combination of \( \langle \{y_1,y_2,\ldots,y_n\} \rangle \), as in Equation \( \text{ref}[eq:9.1.3] \). In this case we say that Equation \( \text{ref}[eq:9.1.3] \) is the **general solution** of \( \langle Ly=0 \rangle \) on \( \langle (a,b) \rangle \).

It can be shown (Exercises 9.1.14 and 9.1.15) that if the equation \( \langle Ly=0 \rangle \) is normal on \( \langle (a,b) \rangle \) then it has infinitely many fundamental sets of solutions on \( \langle (a,b) \rangle \). The next definition will help to identify fundamental sets of solutions of \( \langle Ly=0 \rangle \).

We say that \( \langle \{y_1,y_2,\ldots,y_n\} \rangle \) is **linearly independent** on \( \langle (a,b) \rangle \) if the only constants \( \langle c_1 \rangle, \langle c_2 \rangle, \ldots, \langle c_n \rangle \) such that
\[
\langle c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)=0, \quad a<x<b, \rangle
\]
are \( \langle c_1=c_2=\cdots=c_n=0 \rangle \). If Equation \( \text{ref}[eq:9.1.4] \) holds for some set of constants \( \langle c_1 \rangle, \langle c_2 \rangle, \ldots, \langle c_n \rangle \) that are not all zero, then \( \langle \{y_1,y_2,\ldots,y_n\} \rangle \) is **linearly dependent** on \( \langle (a,b) \rangle \).

The next theorem is analogous to Theorem 5.1.3.

**Theorem**

If \( \langle Ly=0 \rangle \) is normal on \( \langle (a,b) \rangle \), then a set \( \langle \{y_1,y_2,\ldots,y_n\} \rangle \) of \( \langle n \rangle \) solutions of \( \langle Ly=0 \rangle \) on \( \langle (a,b) \rangle \) is a fundamental set if and only if it is linearly independent on \( \langle (a,b) \rangle \).

**Example**

The equation
is normal and has the solutions \(y_1=x^2\), \(y_2=x^3\), and \(y_3=1/x\) on \((-\infty,0)\) and \((0,\infty)\). Show that \(\{y_1, y_2, y_3\}\) is linearly independent on \((-\infty, 0)\) and \((0,\infty)\). Then find the general solution of Equation \ref{eq:9.1.5} on \((-\infty, 0)\) and \((0,\infty)\).

**Solution**

Suppose

\[c_1x^2+c_2x^3+{c_3}/x=0\]

on \((0,\infty)\). We must show that \(c_1=c_2=c_3=0\). Differentiating Equation \ref{eq:9.1.6} twice yields the system

\[
\begin{array}{rr}
{c_1x^2+c_2x^3+{c_3}/x} &{= 0} \\
{2c_1x+3c_2x^2-{c_3}/x^2} &{=0} \\
{2c_1+6c_2x + {2c_3}/x^3} &{=0}.
\end{array}
\]

If Equation \ref{eq:9.1.7} holds for all \(x\) in \((0,\infty)\), then it certainly holds at \(x=1\); therefore,

\[
\begin{array}{rr}
{\phantom{2}c_1+\phantom{3}c_2+\phantom{2}c_3} &{= 0} \\
{2c_1+3c_2-\phantom{2}c_3} &{= 0} \\
{2c_1+6c_2+2c_3} &{=0}.
\end{array}
\]

By solving this system directly, you can verify that it has only the trivial solution \(c_1=c_2=c_3=0\); however, for our purposes it is more useful to recall from linear algebra that a homogeneous linear system of \(n\) equations in \(n\) unknowns has only the trivial solution if its determinant is nonzero. Since the determinant of Equation \ref{eq:9.1.8} is

\[
\left|\begin{array}{rrr}1&1&1\hspace{1cm}\left|\begin{array}{rrr}1&0&0\hspace{1cm}\left|\begin{array}{rrr}1&0&0\end{array}\right|= 12, \nonumber\end{array}\nonumber\right|
\right|
\]

it follows that Equation \ref{eq:9.1.8} has only the trivial solution, so \(\{y_1, y_2, y_3\}\) is linearly independent on \((0,\infty)\). Now Theorem \((\text{PageIndex} \{2\})\) implies that

\[y = c_1x^2+c_2x^3+{c_3}/x\]

is the general solution of Equation \ref{eq:9.1.5} on \((0,\infty)\). To see that this is also true on \((-\infty,0)\), assume that Equation \ref{eq:9.1.6} holds on \((-\infty,0)\). Setting \(x=1\) in Equation \ref{eq:9.1.7} yields

\[
\begin{array}{rrr}
{\phantom{-2}c_1-\phantom{3}c_2-\phantom{2}c_3} &{=0} \\
{-2c_1+3c_2-\phantom{2}c_3} &{=0} \\
{2c_1-6c_2-2c_3} &{=0}.
\end{array}
\]

Since the determinant of this system is

\[
\left|\begin{array}{rrr}1&-1&-1\hspace{1cm}\left|\begin{array}{rrr}1&0&0\hspace{1cm}\left|\begin{array}{rrr}1&0&0\end{array}\right|=-12, \nonumber\end{array}\nonumber\right|
\right|
\]

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it follows that \(c_1=c_2=c_3=0\); that is, \(\{y_1,y_2,y_3\}\) is linearly independent on \((\infty,0)\).

Example \(\PageIndex{2}\)

The equation
\[
\label{eq:9.1.9} y^{(4)}+y'''-7y''-y'+6y=0
\]
is normal and has the solutions \(y_1=e^x\), \(y_2=e^{-x}\), \(y_3=e^{2x}\), and \(y_4=e^{-3x}\) on \((\infty,\infty)\).

(Verify.) Show that \(\{y_1,y_2,y_3,y_4\}\) is linearly independent on \((\infty,\infty)\). Then find the general solution of Equation \ref{eq:9.1.9}.

Solution

Suppose \(c_1\), \(c_2\), \(c_3\), and \(c_4\) are constants such that
\[
\label{eq:9.1.10} c_1e^x+c_2e^{-x}+c_3e^{2x}+c_4e^{-3x}=0
\]
for all \(x\). We must show that \(c_1=c_2=c_3=c_4=0\). Differentiating Equation \ref{eq:9.1.10} three times yields the system
\[
\label{eq:9.1.11} \begin{array}{rcl}
c_1e^x+c_2e^{-x}+c_3e^{2x}+c_4e^{-3x}&=&0 \\
c_1e^x-c_2e^{-x}+2c_3e^{2x}-3c_4e^{-3x}&=&0 \\
c_1e^x+c_2e^{-x}+4c_3e^{2x}+9c_4e^{-3x}&=&0 \\
c_1e^x-c_2e^{-x}+8c_3e^{2x}-27c_4e^{-3x}&=&0
\end{array}
\]
If Equation \ref{eq:9.1.11} holds for all \(x\), then it certainly holds for \(x=0\). Therefore
\[
\begin{array}{rcl}
c_1&+&c_2 \\
c_3&+&c_4 \\
c_1+c_2 +4c_3 &+&c_4 \\
c_1+c_2 +8c_3 &+&27c_4
\end{array}&=&0\ \& 0\ \& 0\ \& 0
\]
The determinant of this system is
\[
\label{eq:9.1.12} \begin{array}{rcl}
&=&-2\left|\begin{array}{rr}3&8 \\
0&-24\end{array}\right|=-240
\end{array}
\]
so the system has only the trivial solution \(c_1=c_2=c_3=c_4=0\). Now Theorem \(\PageIndex{2}\) implies that
\[
y=e^x+c_2e^{-x}+c_3e^{2x}+c_4e^{-3x}
\]
is the general solution of Equation \ref{eq:9.1.9}.
The Wronskian

We can use the method used in Examples (PageIndex{1}) and (PageIndex{2}) to test \( n \) solutions \( \{y_1,y_2,\ldots,y_n\} \) of any \( n \)th order equation \( (Ly=0) \) for linear independence on an interval \( ((a,b)) \) on which the equation is normal. Thus, if \( (c_1), (c_2), \ldots, (c_n) \) are constants such that

\[
[c_1y_1+c_2y_2+\cdots+c_ny_n=0, \quad a<b, \nonumber]
\]

then differentiating \( (n-1) \) times leads to the \((n\times n)\) system of equations

\[
\begin{array}{rcl}
  c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)=0 \\
  c_1y_1'(x)+c_2y_2'(x)+\cdots+c_ny_n'(x)=0 \\
  \vdots \\
  c_1y_1^{(n-1)}(x)+c_2y_2^{(n-1)}(x)+\cdots+c_ny_n^{(n-1)}(x) =0
\end{array}
\]

for \( (c_1), (c_2), \ldots, (c_n) \). For a fixed \( (x) \), the determinant of this system is

\[
[W(x)=\left|\begin{array}{cccc}
  y_1(x) & y_2(x) & \cdots & y_n(x) \\
  y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\
  \vdots & \vdots & \ddots & \vdots \\
  y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x)
\end{array}\right|, \quad a<b.]
\]

We call this determinant the Wronskian of \( \{y_1,y_2,\ldots,y_n\} \). If \( (W(x)\ne0) \) for some \( (x) \) in \( ((a,b)) \) then the system Equation \ref{eq:9.1.13} has only the trivial solution \( (c_1=c_2=\cdots=c_n=0) \), and Theorem (PageIndex{2}) implies that

\[
y=c_1y_1+c_2y_2+\cdots+c_ny_n
\]

is the general solution of \( (Ly=0) \) on \( ((a,b)) \).

The next theorem generalizes Theorem 5.1.4. The proof is sketched in Exercises 9.1.17-9.1.20.

Theorem (PageIndex{3}) Abel's Formula

Suppose the homogeneous linear \( (n)\)th order equation

\[
\begin{align*}
\text{P}_0(x)y^{(n)}+\text{P}_1(x)y^{(n-1)}+\cdots+\text{P}_n(x)y &= 0 \\
\end{align*}
\]

is normal on \( ((a,b)) \) let \( (y_1, y_2, \ldots, y_n) \) be solutions of Equation \ref{eq:9.1.14} on \( ((a,b),) \) and let \( (x_0) \) be in \( ((a,b)). \) Then the Wronskian of \( \{y_1,y_2,\ldots,y_n\} \) is given by

\[
W(x)=W(x_0)\exp\left(-\int^{x}_{x_0}{P_1(t) \over P_0(t)} \, dt \right), \quad a<b.
\]

Therefore, either \( (W) \) has no zeros in \( ((a,b)) \) or \( (W\equiv0) \) on \( ((a,b)). \)

Formula Equation \ref{eq:9.1.15} is Abel's formula.

The next theorem is analogous to Theorem 5.1.6.
Theorem \(\PageIndex{4}\)

Suppose \(Ly=0\) is normal on \((a,b)\) and let \(\{y_1, y_2, \ldots, y_n\}\) be \(n\) solutions of \(Ly=0\) on \((a,b)\). Then the following statements are equivalent (\(\ast\)) that is, they are either all true or all false:

a. The general solution of \(Ly=0\) on \((a,b)\) is \(y=c_1y_1+c_2y_2+\cdots+c_ny_n\).

b. \(\{y_1, y_2, \ldots, y_n\}\) is a fundamental set of solutions of \(Ly=0\) on \((a,b)\).

c. \(\{y_1, y_2, \ldots, y_n\}\) is linearly independent on \((a,b)\).

d. The Wronskian of \(\{y_1, y_2, \ldots, y_n\}\) is nonzero at some point in \((a,b)\).

e. The Wronskian of \(\{y_1, y_2, \ldots, y_n\}\) is nonzero at all points in \((a,b)\).

Example \(\PageIndex{3}\)

In Example \(\PageIndex{1}\) we saw that the solutions \(y_1=x^2\), \(y_2=x^3\), and \(y_3=1/x\) of

\[x^3y'''-x^2y''-2xy'+6y=0\]

are linearly independent on \((-\infty,0)\) and \((0,\infty)\). Calculate the Wronskian of \(\{y_1,y_2,y_3\}\).

Solution

If \(x\ne0\), then

\[
W(x)=\left|\begin{array}{ccc}x^2 & x^3 & \frac{1}{x} \\
2x & 3x^2 & -\frac{1}{x^2} \\
2 & 6x & \frac{2}{x^3}\end{array}\right|
\]

where we factored \((x^2), (x),\) and \((2)\) out of the first, second, and third rows of \(W(x)\), respectively. Adding the second row of the last determinant to the first and third rows yields

\[
W(x)=2x^3\left|\begin{array}{cc}3 & 4x \\
3 & 6x\end{array}\right| = 12x
\]

Therefore \(W(x)\ne0\) on \((-\infty,0)\) and \((0,\infty)\).

Example \(\PageIndex{4}\)

In Example \(\PageIndex{2}\) we saw that the solutions \(y_1=e^x\), \(y_2=e^{-x}\), \(y_3=e^{2x}\), and \(y_4=e^{-3x}\) of

\[y^{(4)}+y'''-7y''-y'+6y=0\]

are linearly independent on every open interval. Calculate the Wronskian of \(\{y_1,y_2,y_3,y_4\}\).

Solution
For all \( x \),
\[
W(x) = \begin{vmatrix}
  e^x & e^{-x} & e^{2x} & e^{-3x} \\
  e^x & -e^{-x} & 2e^{2x} & -3e^{-3x} \\
  e^x & e^{-x} & 4e^{2x} & 9e^{-3x} \\
  e^x & -e^{-x} & 8e^{2x} & -27e^{-3x}
\end{vmatrix},
\]

Factoring the exponential common factor from each row yields
\[
W(x) = e^{-x} \begin{vmatrix}
  1 & 1 & 1 & 1 \\
  1 & -1 & 2 & -3 \\
  1 & 1 & 4 & 9 \\
  1 & -1 & 8 & -27
\end{vmatrix} = 240e^{-x},
\]
from Equation \ref{eq:9.1.12}.

Note
Under the assumptions of Theorem \( \PageIndex{4} \), it isn’t necessary to obtain a formula for \( W(x) \). Just evaluate \( W(x) \) at a convenient point in \((a, b)\), as we did in Examples \( \PageIndex{1} \) and \( \PageIndex{2} \).

Theorem \( \PageIndex{5} \)
Suppose \( c \) is in \((a, b)\) and \( \alpha_{1}, \alpha_{2}, \ldots \) are real numbers, not all zero. Under the assumptions of Theorem \ref{thm:10.3.3}, suppose \( y_{1} \) and \( y_{2} \) are solutions of Equation \ref{eq:5.1.35} such that
\[
\alpha_{i}(c)+ y_{i}'(c)+\cdots+y_{i}^{(n-1)}(c)=0, \quad 1\le i\le n.
\]
Then \( \{y_{1}, y_{2}, \ldots, y_{n}\} \) isn’t linearly independent on \((a, b)\).

Proof
Since \( \alpha_{1}, \alpha_{2}, \ldots, \alpha_{n} \) are not all zero, Equation \ref{eq:9.1.14} implies that
\[
\begin{vmatrix}
  y_{1}(c) & y_{1}'(c) & \cdots & y_{1}^{(n-1)}(c) \\
  y_{2}(c) & y_{2}'(c) & \cdots & y_{2}^{(n-1)}(c) \\
  \vdots & \vdots & \ddots & \vdots \\
  y_{n}(c) & y_{n}'(c) & \cdots & y_{n}^{(n-1)}(c)
\end{vmatrix} = 0,
\]
so
\[
\begin{vmatrix}
  y_{1}(c) & y_{2}(c) & \cdots & y_{n}(c) \\
  y_{1}'(c) & y_{2}'(c) & \cdots & y_{n}'(c) \\
  \vdots & \vdots & \ddots & \vdots \\
  y_{1}^{(n-1)}(c) & y_{2}^{(n-1)}(c) & \cdots & y_{n}^{(n-1)}(c)
\end{vmatrix} = 0.
\]
and Theorem \( \PageIndex{4} \) implies the stated conclusion.
General Solution of a Nonhomogeneous Equation

The next theorem is analogous to Theorem 5.3.2. It shows how to find the general solution of \((Ly=F)\) if we know a particular solution of \((Ly=F)\) and a fundamental set of solutions of the complementary equation \((Ly=0)\).

Theorem \(\PageIndex{6}\)

The probabilities assigned to events by a distribution function on a sample space are given by.

**Proof**

Suppose \((Ly=F)\) is normal on \(((a,b))\). Let \((y_p)\) be a particular solution of \((Ly=F)\) on \(((a,b))\) and let \((\{y_1,y_2,\ldots,y_n\})\) be a fundamental set of solutions of the complementary equation \((Ly=0)\) on \(((a,b))\). Then \((y)\) is a solution of \((Ly=F)\) on \(((a,b))\) if and only if

\[
y = y_p + c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,
\]

where \((c_1, c_2, \ldots, c_n)\) are constants.

The next theorem is analogous to Theorem 5.3.2.

Theorem \(\PageIndex{7}\) The Principle of Superposition

Suppose for each \((i=1, 2, \ldots, r)\), the function \((y_{p_i})\) is a particular solution of \((Ly=F_i)\) on \(((a,b))\). Then

\[
y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_r}
\]

is a particular solution of

\[
Ly = F_1(x) + F_2(x) + \cdots + F_r(x)
\]

on \(((a,b))\).

We’ll apply Theorems \(\PageIndex{6}\) and \(\PageIndex{7}\) throughout the rest of this chapter.