In this section we consider eigenvalue problems of the form

\[
P_0(x)y'' + P_1(x)y' + P_2(x)y + \lambda R(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0,
\]

where

\[
[B_1(y)] = \alpha y(a) + \beta y'(a) \quad \text{and} \quad [B_2(y)] = \rho y(b) + \delta y'(b).
\]
As in Section 13.1, \(\alpha\), \(\beta\), \(\rho\), and \(\delta\) are real numbers, with

\[
\alpha^2 + \beta^2 > 0 \quad \text{and} \quad \rho^2 + \delta^2 > 0, \nonumber
\]

\(P_0\), \(P_1\), \(P_2\), and \(R\) are continuous, and \((P_0')\) and \((R')\) are positive on \([a,b]\).

We say that \(\lambda\) is an eigenvalue of Equation \ref{eq:13.2.1} if Equation \ref{eq:13.2.1} has a nontrivial solution \(y\).

In this case, \(y\) is an eigenfunction associated with \(\lambda\), or a \(\lambda\)-eigenfunction. Solving the eigenvalue problem means finding all eigenvalues and associated eigenfunctions of Equation \ref{eq:13.2.1}.

Example \(\PageIndex{1}\)

Solve the eigenvalue problem

\[
\label{eq:13.2.2} y'+3y'+2y+\lambda y=0, \quad y(0)=0, \quad y(1)=0. \nonumber
\]

**Solution**

The characteristic equation of Equation \ref{eq:13.2.2} is

\[
[r^2+3r+2+\lambda=0, \nonumber]\]

with zeros

\[
[r_1=\frac{-3+\sqrt{1-4\lambda}}{2} \quad \text{and} \quad r_2=\frac{-3-\sqrt{1-4\lambda}}{2}. \nonumber]\]

If \(\lambda < 1/4\) then \(r_1\) and \(r_2\) are real and distinct, so the general solution of the differential equation in Equation \ref{eq:13.2.2} is

\[
y=c_1e^{r_1t}+c_2e^{r_2t}. \nonumber\]

The boundary conditions require that

\[
\begin{aligned}
c_1 + c_2 &= 0, \\
c_1e^{r_1} + c_2e^{r_2} &= 0.
\end{aligned}\]

Since the determinant of this system is \(e^{r_2} - e^{r_1} \neq 0\), the system has only the trivial solution. Therefore \(\lambda\) isn’t an eigenvalue of Equation \ref{eq:13.2.2}.

If \(\lambda = 1/4\) then \(r_1 = r_2 = -3/2\), so the general solution of Equation \ref{eq:13.2.2} is

\[
y=e^{-3x/2}(c_1+c_2x). \nonumber\]

The boundary condition \(y(0)=0\) requires that \(c_1=0\), so \(y=c_2x e^{-3x/2}\) and the boundary condition \(y(0)=0\) requires that \(c_2=0\). Therefore \(\lambda=1/4\) isn’t an eigenvalue of Equation \ref{eq:13.2.2}.
If $\lambda > 1/4$ then

\[ r_1 = -\frac{3}{2} + i\omega \quad \text{and} \quad r_2 = -\frac{3}{2} - i\omega, \]

with

\[ \omega = \frac{\sqrt{4\lambda - 1}}{2} \quad \text{or equivalently} \quad \lambda = \frac{1 + 4\omega^2}{4}. \]

In this case the general solution of the differential equation in Equation \ref{eq:13.2.2} is

\[ y = e^{-3x/2}(c_1 \cos \omega x + c_2 \sin \omega x). \]

The boundary condition $y(0) = 0$ requires that $(c_1 = 0)$, so $y = c_2 e^{-3x/2} \sin \omega x$, which holds with $(c_2 \neq 0)$ if and only if $(\omega = n\pi)$, where $(n)$ is an integer. We may assume that $(n)$ is a positive integer. (Why?). From Equation \ref{eq:13.2.3}, the eigenvalues are $(\lambda_n = (1 + 4n^2\pi^2)/4)$, with associated eigenfunctions

\[ y_n = e^{-3x/2}\sin n\pi x, \quad n = 1, 2, 3, \ldots. \]

Example \(\PageIndex{2}\)

Solve the eigenvalue problem

\[ x^2y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0. \]

Solution

If $(\lambda = 0)$, the differential equation in Equation \ref{eq:13.2.4} reduces to $(x(xy')' = 0)$, so $(xy = c_1 \ln x)$,

\[ y = \frac{c_1 \ln x + c_2}{x}, \quad \text{and} \quad y = c_1 \ln x + c_2. \]

The boundary condition $(y(1) = 0)$ requires that $(c_1 = 0)$, so $(y = c_1 \ln x)$. The boundary condition $(y(2) = 0)$ requires that $(c_1 \ln 2 = 0)$, so $(c_1 = 0)$. Therefore zero isn’t an eigenvalue of Equation \ref{eq:13.2.4}.

If $(\lambda < 0)$, we write $(\lambda = -k^2)$ with $(k > 0)$, so Equation \ref{eq:13.2.4} becomes

\[ x^2y'' + xy' - k^2y = 0, \]

an Euler equation (Section 7.4) with indicial equation

\[ r^2 - k^2 = (r-k)(r+k) = 0. \]

Therefore

\[ y = c_1 x^k + c_2 x^{-k}. \]
The boundary conditions require that

\[
\begin{aligned}
\phantom{2^{k}}c_{1}+\phantom{2^{-k}}c_{2}&=0 \\
2^{k}c_{1}+2^{-k}c_{2}&=0.
\end{aligned}
\]

Since the determinant of this system is \(2^{k}+2^{-k}\neq 0\), \((c_{1}=c_{2}=0)\). Therefore Equation \ref{eq:13.2.4} has no negative eigenvalues.

If \((\lambda>0)\) we write \((\lambda=k^{2}\{2\})\) with \((k>0)\). Then Equation \ref{eq:13.2.4} becomes

\[
[x^{2}\{2\}y''+xy' +k^{2}y=0, \nonumber 
\]

an Euler equation with indicial equation

\[
[r^{2}+k^{2}=(r-ik)(r+ik)=0, \nonumber 
\]

so

\[
y=c_{1}\cos(k\ln x)+c_{2}\sin(k\ln x). \nonumber 
\]

The boundary condition \((y(1)=0)\) requires that \((c_{1}=0)\). Therefore \((y=c_{2}\sin(k\ln x))\). This holds with \((c_{2}\neq 0)\) if and only if \((k=n\pi/\ln(2))\), where \((n)\) is a positive integer. Hence, the eigenvalues of Equation \ref{eq:13.2.4} are \((\lambda_{n}=n\pi^2/\ln^2(2))\), with associated eigenfunctions

\[
y_{n}=\sin\left(\frac{n\pi}{\ln(2)}\ln x\right), \quad n=1,2,3,\ldots. \nonumber 
\]

For theoretical purposes, it is useful to rewrite the differential equation in Equation \ref{eq:13.2.1} in a different form, provided by the next theorem.

**Theorem** \(\PageIndex{1}\)

If \((P_{0},P_{1},P_{2},\) and \(R)\) are continuous and \((P_{0}\) and \(R)\) are positive on a closed interval \([a,b]\), then the equation

\[
P_{0}(x)y''+P_{1}(x)y'+P_{2}(x)y+\lambda R(x)y=0, \nonumber
\]

can be rewritten as

\[
(p(x)y')'+q(x)y+\lambda r(x)y=0, \nonumber
\]

where \((p), (p'), (q)\) and \((r)\) are continuous and \((p)\) and \((r)\) are positive on \([a,b]\).

**Proof**

We begin by rewriting Equation \ref{eq:13.2.5} as

\[
y''+u(x)y'+v(x)y+\lambda R_{1}(x)y=0, \nonumber
\]
with \(u=P_{1}/P_{0}\), \(v=P_{2}/P_{0}\), and \(R_{1}=R/P_{0}\). (Note that \(R_{1}\) is positive on \([a,b]\)). Now let \(p(x)=e^{U(x)}\), where \(U\) is any antiderivative of \((u')\). Then \(p\) is positive on \([a,b]\) and, since \(U'=u\),

\[
\label{eq:13.2.8} p'(x)=p(x)u(x)
\]

is continuous on \([a,b]\). Multiplying Equation \ref{eq:13.2.7} by \(p(x)\) yields

\[
\label{eq:13.2.9} p(x)y''+p(x)u(x)y'+p(x)v(x)y+\lambda p(x)R_{1}(x)y=0.
\]

Since \(p\) is positive on \([a,b]\), this equation has the same solutions as Equation \ref{eq:13.2.5}. From Equation \ref{eq:13.2.8},

\[
[(p(x)y')'=p(x)y''+p'(x)y'=p(x)y''+p(x)u(x)y', \nonumber \]

so Equation \ref{eq:13.2.9} can be rewritten as in Equation \ref{eq:13.2.6}, with \(q(x)=p(x)v(x)\) and \(r(x)=p(x)R_{1}(x)\). This completes the proof.

It is to be understood throughout the rest of this section that \(p\), \(q\), and \(r\) have the properties stated in Theorem \ref{PageIndex{1}}. Moreover, whenever we write \((Ly)\) in a general statement, we mean

\[
[Ly=(p(x)y')'+q(x)y. \nonumber \]

The differential equation Equation \ref{eq:13.2.6} is called a Sturm-Liouville equation, and the eigenvalue problem

\[
\label{eq:13.2.10} (p(x)y')'+q(x)y+\lambda r(x)y=0, \quad B_{1}(y)=0, \quad B_{2}(y)=0,
\]

which is equivalent to Equation \ref{eq:13.2.1}, is called a Sturm-Liouville problem.

Example \ref{PageIndex{3}}

Rewrite the eigenvalue problem

\[
\label{eq:13.2.11} y''+3y'+(2+\lambda)y=0, \quad y(0)=0, \quad y(1)=0
\]

of Theorem \ref{PageIndex{1}} as a Sturm-Liouville problem.

**Solution**

Comparing Equation \ref{eq:13.2.11} to Equation \ref{eq:13.2.7} shows that \((u(x)=3)\), so we take \((U(x)=3x)\) and \((p(x)=e^{3x})\). Multiplying the differential equation in Equation \ref{eq:13.2.11} by \(e^{3x}\) yields

\[
[e^{3x}(y''+3y')+2e^{3x}y+\lambda e^{3x}y=0. \nonumber \]

Since
\[ e^{3x}(y''+3y') = (e^{3x}y')', \text{ nonumber} \]

Equation \ref{eq:13.2.11} is equivalent to the Sturm–Liouville problem

\[ (e^{3x}y')' + 2e^{3x}y + \lambda e^{3x}y = 0, \quad y(0) = 0, \quad y(1) = 0. \]

Example \(\PageIndex{4}\)

Rewrite the eigenvalue problem

\[ x^2y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0 \]

of Theorem \(\PageIndex{2}\) as a Sturm-Liouville problem.

**Solution**

Dividing the differential equation in Equation \ref{eq:13.2.13} by \(x^2\) yields

\[ y'' + \frac{1}{x}y' + \frac{\lambda}{x^2}y = 0. \text{ nonumber} \]

Comparing this to Equation \ref{eq:13.2.7} shows that \(u(x) = 1/x\), so we take \(U(x) = \ln x\) and \(p(x) = e^{\ln x} = x\).

Multiplying the differential equation by \(x\) yields

\[ xy'' + y' + \frac{\lambda}{x}y = 0. \text{ nonumber} \]

Since

\[ xy'' + y' = (xy')', \text{ nonumber} \]

Equation \ref{eq:13.2.13} is equivalent to the Sturm–Liouville problem

\[ (xy')' + \frac{\lambda}{x}y = 0, \quad y(1) = 0, \quad y(2) = 0. \]

Problems 1–4 of Section 11.1 are Sturm–Liouville problems. (Problem 5 isn’t, although some authors use a definition of *Sturm-Liouville problem* that does include it.) We were able to find the eigenvalues of Problems 1–4 explicitly because in each problem the coefficients in the boundary conditions satisfy \((\alpha\beta = 0)\) and \((\rho\delta = 0)\); that is, each boundary condition involves either \(y\) or \(y'\), but not both. If this isn’t true then the eigenvalues can’t in general be expressed exactly by simple formulas; rather, approximate values must be obtained by numerical solution of equations derived by requiring the determinants of certain \(2\times 2\) systems of homogeneous equations to be zero. To apply the numerical methods effectively, graphical methods must be used to determine approximate locations of the zeros of these determinants. Then the zeros can be computed accurately by numerical methods.

Example \(\PageIndex{5}\)

Solve the Sturm–Liouville problem
\[ \text{Solution} \]

If \( \lambda = 0 \), the differential equation in Equation \ref{eq:13.2.15} reduces to \( y'' = 0 \), with general solution \( y = c_1 + c_2 x \). The boundary conditions require that

\[
\begin{aligned}
c_1 + c_2 &= 0 \\
c_1 + 4c_2 &= 0,
\end{aligned}
\]

so \( c_1 = c_2 = 0 \). Therefore zero isn’t an eigenvalue of Equation \ref{eq:13.2.15}.

If \( \lambda < 0 \), we write \( \lambda = -k^2 \) where \( k > 0 \), and the differential equation in Equation \ref{eq:13.2.15} becomes \( y'' - k^2 y = 0 \), with general solution

\[ y = c_1 \cosh kx + c_2 \sinh kx, \]

so

\[ y' = k(c_1 \sinh kx + c_2 \cosh kx). \]

The boundary conditions require that

\[
\begin{aligned}
c_1 + kc_2 &= 0 \\
(\cosh k + 3k \sinh k)c_1 + (\sinh k + 3k \cosh k)c_2 &= 0,
\end{aligned}
\]

The determinant of this system is

\[
D_N(k) = \left| \begin{array}{cc} 1 & k \\ \cosh k + 3k \sinh k & \sinh k + 3k \cosh k \end{array} \right| = (1 - 3k^2) \sinh k + 2k \cosh k.
\]

Therefore the system Equation \ref{eq:13.2.17} has a nontrivial solution if and only if \( D_N(k) = 0 \) or, equivalently,

\[ \tanh k = -\frac{2k}{1 - 3k^2} \]

The graph of the right side (Figure \PageIndex{1}) has a vertical asymptote at \( k = \frac{1}{\sqrt{3}} \). Since the two sides have different signs if \( k < \frac{1}{\sqrt{3}} \), this equation has no solution in \( (0, 1/\sqrt{3}) \). Figure \PageIndex{1} shows the graphs of the two sides of Equation \ref{eq:13.2.18} on an interval to the right of the vertical asymptote, which is indicated by the dashed line. You can see that the two curves intersect near \( k = 1.2 \). Given this estimate, you can use Newton’s to compute \( k_0 \) more accurately. We computed \( k_0 \approx 1.2193951 \). Therefore \( (-k_0)^2 \approx 1.2587483 \) is an eigenvalue of Equation \ref{eq:13.2.15}. From Equation \ref{eq:13.2.16} and the first equation in Equation \ref{eq:13.2.17},

\[ y_0 = k_0 \cosh k_0 x - \sinh k_0 x. \]
If \( \lambda > 0 \) we write \( \lambda = k^2 \) where \( k > 0 \), and differential equation in Equation \( \text{ref} \{eq:13.2.15 \} \) becomes
\[ y'' + k^2 y = 0, \]
with general solution
\[ y = \cos kx + c_2 \sin kx, \]
so
\[ y' = k(-c_1 \sin kx + c_2 \cos kx). \]
The boundary conditions require that
\[ \begin{array}{c}
\cos k x + c_2 \sin k x = 0 \\
(k \cos k - 3k \sin k)c_1 + (k \sin k + 3k \cos k)c_2 = 0.
\end{array} \]
The determinant of this system is
\[ D_P(k) = (1 + 3k^2) \sin k + 2k \cos k. \]
The system Equation \( \text{ref} \{eq:13.2.20 \} \) has a nontrivial solution if and only if \( D_P(k) = 0 \) or, equivalently,
\[ \tan k = -\frac{2k}{1 + 3k^2}. \]
Figure \( \text{PageIndex} \{2 \} \) shows the graphs of the two sides of this equation. You can see from the figure that the graphs intersect at infinitely many points \( k_n \approx n \pi \) \( (n = 1, 2, 3, \ldots) \), where the error in this approximation approaches zero as \( n \to \infty \). Given this estimate, you can use Newton’s method to compute \( k_n \) more accurately.

We computed
\[ \begin{aligned}
k_1 &\approx 2.9256856, \\
k_2 &\approx 6.1765914, \\
k_3 &\approx 9.3538959, \\
k_4 &\approx 12.5132570.
\end{aligned} \]
The estimates of the corresponding eigenvalues \( \lambda_n = k_n^2 \) are
From Equation \ref{eq:13.2.19} and the first equation in Equation \ref{eq:13.2.20},
\[y_{n}=k_{n}\cos k_{n}x-\sin k_{n}x\nonumber\]
is an eigenfunction associated with \(\lambda_{n}\)

![Figure](https://example.com/fig.png)

Since the differential equations in Equation \ref{eq:13.2.12} and Equation \ref{eq:13.2.14} are more complicated than those in Equation \ref{eq:13.2.11} and Equation \ref{eq:13.2.13} respectively, what is the point of Theorem \(\PageIndex{1}\)? The point is this: to solve a specific problem, it may be better to deal with it directly, as we did in Examples \(\PageIndex{1}\) and \(\PageIndex{2}\); however, we’ll see that transforming the general eigenvalue problem Equation \ref{eq:13.2.1} to the Sturm–Liouville problem Equation \ref{eq:13.2.10} leads to results applicable to all eigenvalue problems of the form Equation \ref{eq:13.2.1}.

Theorem \(\PageIndex{2}\)

If
\[Ly=(p(x)y')'+q(x)y\nonumber\]
and \((u)\) and \((v)\) are twice continuously functions on \([a,b]\) that satisfy the boundary conditions \((B\_1(y)=0)\) and \((B\_2(y)=0,0)\) then
\[
\int_{a}^{b}[u(x)Lv(x)-v(x)Lu(x)]\,dx=0.\]

**Proof**

Integration by parts yields

\[
\begin{aligned}
\lambda_1 &\approx 8.5596361, \\
\lambda_2 &\approx 38.1502809, \\
\lambda_3 &\approx 87.4953676, \\
\lambda_4 &\approx 156.5815998.
\end{aligned}
\]
\[
\begin{aligned}
&\int_{a}^{b}[u(x)Lv(x)-v(x)Lu(x)]\,dx = p(x)[u(x)v'(x)-u'(x)v(x)]|_{a}^{b} \\
&\text{and} \\
&\int_{a}^{b}p(x)[u'(x)v'(x)-u'(x)v'(x)]\,dx = 0.
\end{aligned}
\]

Since the last integral equals zero,
\[
\int_{a}^{b}[u(x)Lv(x)-v(x)Lu(x)]\,dx = p(x)[u(x)v'(x)-u'(x)v'(x)]|_{a}^{b}.
\]

By assumption, \(B_{1}(u)=B_{1}(v)=0\) and \(B_{2}(u)=B_{2}(v)=0\). Therefore
\[
\begin{aligned}
&\alpha u(a)+\beta u'(a)=0 \\
&\alpha v(a)+\beta v'(a)=0 \\
&\rho u(b)+\delta u'(b)=0 \\
&\rho v(b)+\delta v'(b)=0.
\end{aligned}
\]

Since \(\alpha^{2}+\beta^{2}>0\) and \(\rho^{2}+\delta^{2}>0\), the determinants of these two systems must both be zero; that is,
\[\[u(a)v'(a)-u'(a)v(a)=u(b)v'(b)-u'(b)v(b)=0. \quad \text{nonumber}\]

This and Equation \ref{eq:13.2.22} imply Equation \ref{eq:13.2.21}, which completes the proof.

The next theorem shows that a Sturm–Liouville problem has no complex eigenvalues.

Theorem \(\PageIndex{3}\)

If \(\lambda=p+qi\) with \(q\neq0\) then the boundary value problem
\[
\begin{aligned}
&Ly+(p+iq)r(x)y=0, \\
&B_{1}(y)=0, \\
&B_{2}(y)=0
\end{aligned}
\]

has only the trivial solution.

Proof

For this theorem to make sense, we must consider complex-valued solutions of
\[
\begin{aligned}
&\text{label{eq:13.2.23}} \quad Ly+(p+iq)r(x,y)y=0.\]
\]

If \(y=u+iv\) where \(u\) and \(v\) are real-valued and twice differentiable, we define \(y'=u'+iv'\) and \(y''=u''+iv''\). We say that \(y\) is a solution of Equation \ref{eq:13.2.23} if the real and imaginary parts of the left side of Equation \ref{eq:13.2.23} are both zero. Since \(\lambda=(p(x)'y')+q(x)y\) and \(\lambda(p), \quad \lambda(q), \quad \lambda(r)\) are real-valued,
\[
\begin{aligned}
&\text{label{aligned}} \quad Ly+(p+iq)r(x,y)y=0. \\
&\text{and} \\
&Lu+(r(x)(pu-qv)+i[Ly+r(x)(qu+pv)],\text{end{aligned}}\]

so \(Ly+(\lambda y)=0\) if and only if
\[
\begin{aligned}
&\text{label{aligned}} \quad Lu+r(x)(pu-qv)&=0. \\
&Lv+r(x)(qu+pv)&=0. \quad \text{end{aligned}}\]

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If \(\lambda=p+qi\) with \(q\neq0\) then the boundary value problem
\[
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&B_{1}(y)=0, \\
&B_{2}(y)=0
\end{aligned}
\]

has only the trivial solution.

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\]

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\[
\begin{aligned}
&\text{label{aligned}} \quad Ly+(p+iq)r(x,y)y=0. \\
&\text{and} \\
&Lu+(r(x)(pu-qv)+i[Ly+r(x)(qu+pv)],\text{end{aligned}}\]

so \(Ly+(\lambda y)=0\) if and only if
\[
\begin{aligned}
&\text{label{aligned}} \quad Lu+r(x)(pu-qv)&=0. \\
&Lv+r(x)(qu+pv)&=0. \quad \text{end{aligned}}\]
Multiplying the first equation by \((v)\) and the second by \((u)\) yields
\[
\begin{aligned}
 vLu+r(x)(puv-qv^2) &= 0 \\
 uLv+r(x)(qu^2+puv) &= 0.
\end{aligned}
\]

Subtracting the first equation from the second yields
\[
[uLv-vLu+qr(x)(u^2+v^2)=0, \nonumber \]
so
\[
\int_{a}^{b}[u(x)Lv(x)-v(x)Lu(x)]\,dx+\int_{a}^{b}r(x)[u^2(x)+v^2(x)]\,dx=0. \nonumber \]

Since
\[
[B_1(y)=B_1(u+iv)=B_1(u)+iB_1(v), \nonumber \]
and
\[
[B_2(y)=B_2(u+iv)=B_2(u)+iB_2(v), \nonumber \]
\((B_1(y)=0)\) and \((B_2(y)=0)\) implies that
\[
[B_1(u)=B_1(v)=B_2(u)=B_2(v)=0. \nonumber \]

Therefore Theorem \((\PageIndex{2})\) implies that first integral in Equation \ref{eq:13.2.24} equals zero, so Equation \ref{eq:13.2.24} reduces to
\[
[q\int_{a}^{b}r(x)[u^2(x)+v^2(x)]\,dx = 0. \nonumber \]

Since \((r)\) is positive on \([a,b]\) and \((q\neq 0)\) by assumption, this implies that \((u\equiv 0)\) and \((v\equiv 0)\) on \([a,b]\).

Therefore \((y\equiv 0)\) on \([a,b]\), which completes the proof.

Theorem \((\PageIndex{4})\)

If \((\lambda_1)\) and \((\lambda_2)\) are distinct eigenvalues of the Sturm–Liouville problem
\[
Ly+(\lambda r(x)y=0, \quad B_1(y)=0, \quad B_2(y)=0 \nonumber \]
with associated eigenfunctions \((u)\) and \((v)\) respectively, then
\[
\int_{a}^{b}r(x)u(x)v(x),dx=0. \nonumber \]

Proof

Since \((u)\) and \((v)\) satisfy the boundary conditions in Equation \ref{eq:13.2.25}, Theorem \((\PageIndex{2})\) implies that
\[ \int_{a}^{b} \left[ u(x)Lv(x) - v(x)Lu(x) \right] \, dx = 0. \]

Since \(Lu = \lambda_1 ru\) and \(Lv = \lambda_2 rv\), this implies that

\[ \int_{a}^{b} \left[ (\lambda_1 - \lambda_2) r(x)u(x)v(x) \right] \, dx = 0. \]

Since \(\lambda_1 \neq \lambda_2\), this implies Equation \ref{eq:13.2.26}, which completes the proof.

If \(u\) and \(v\) are any integrable functions on \([a,b]\) and

\[ \int_{a}^{b} r(x)u(x)v(x) \, dx = 0, \]

we say that \(u\) and \(v\) are orthogonal on \([a,b]\) with respect to \(r\).

Theorem 13.1.1 implies the next theorem.

Theorem \ref{PageIndex5}

If \(u \not\equiv 0\) and \(v\) both satisfy

\[ Ly + \lambda r(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0, \]

then \((v = cu)\) for some constant \(c\).

We've now proved parts of the next theorem. A complete proof is beyond the scope of this book.

Theorem \ref{PageIndex6}

The set of all eigenvalues of the Sturm–Liouville problem

\[ Ly + \lambda r(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0 \]

can be ordered as

\[ \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \]

and

\[ \lim_{n \to \infty} \lambda_n = \infty. \]

For each \(n\), if \(y_{\{n\}}\) is an arbitrary \(\{\lambda_1, \ldots, \lambda_n\}\)-eigenfunction, then every \(\{\lambda_1, \ldots, \lambda_n\}\)-eigenfunction is a constant multiple of \(y_{\{n\}}\). If \(m \neq n\), \(y_{\{m\}}\) and \(y_{\{n\}}\) are orthogonal on \([a,b]\) with respect to \(r\) that is,

\[ \int_{a}^{b} r(x)y_{\{m\}}(x)y_{\{n\}}(x) \, dx = 0. \]
You may want to verify Equation \ref{eq:13.2.27} for the eigenfunctions obtained in Examples \(\PageIndex{1}\) and \(\PageIndex{2}\).

In conclusion, we mention the next theorem. The proof is beyond the scope of this book.

**Theorem \(\PageIndex{7}\)**

Let \(\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots\) be the eigenvalues of the Sturm–Liouville problem

\[
Ly + \lambda r(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0,
\]

with associated eigenvectors \(y_1, y_2, \ldots, y_n, \ldots\). Suppose \(f\) is piecewise smooth Theorem 11.2.3 on \([a,b]\). For each \(n\), let

\[
c_n = \frac{\int_a^b r(x)f(x)y_n(x) \, dx}{\int_a^b r(x)y_n^2(x) \, dx}.
\]

Then

\[
\frac{f(x-)+f(x+)}{2} = \sum_{n=1}^\infty c_n y_n(x)
\]

for all \(x\) in the open interval \((a,b)\).