13.2: Sturm-Liouville Problems

In this section we consider eigenvalue problems of the form

\[
P_0(x)y'' + P_1(x)y' + P_2(x)y + \lambda R(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0,
\]

where

\[
B_1(y) = \alpha y(a) + \beta y'(a) \quad \text{and} \quad B_2(y) = \rho y(b) + \delta y'(b).
\]
As in Section 13.1, \( (\alpha), (\beta), (\rho), \) and \( (\delta) \) are real numbers, with

\[
\alpha^2 + \beta^2 > 0 \quad \text{and} \quad \rho^2 + \delta^2 > 0, \quad \text{nonumber}
\]

\( (P_{0}), (P_{1}), (P_{2}), \) and \( (R) \) are continuous, and \( (P_{0}) \) and \( (R) \) are positive on \([a,b])\).

We say that \( (\lambda) \) is an *eigenvalue* of Equation \( \ref{eq:13.2.1} \) if Equation \( \ref{eq:13.2.1} \) has a nontrivial solution \( y \). In this case, \( y \) is an *eigenfunction associated with \( (\lambda) \)*, or a \( (\lambda) \)-eigenfunction. *Solving* the eigenvalue problem means finding all eigenvalues and associated eigenfunctions of Equation \( \ref{eq:13.2.1} \).

Example \( \PageIndex{1} \)

Solve the eigenvalue problem

\[
\label{eq:13.2.2} y' + 3y' + 2y + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0.
\]

**Solution**

The characteristic equation of Equation \( \ref{eq:13.2.2} \) is

\[
[r^2 + 3r + 2 + \lambda = 0, \quad \text{nonumber}]
\]

with zeros

\[
[r_{1} = \frac{-3 + \sqrt{1 - 4\lambda}}{2} \quad \text{and} \quad r_{2} = \frac{-3 - \sqrt{1 - 4\lambda}}{2}. \quad \text{nonumber}]
\]

If \( (\lambda < 1/4) \) then \( (r_{1}) \) and \( (r_{2}) \) are real and distinct, so the general solution of the differential equation in Equation \( \ref{eq:13.2.2} \) is

\[
[y = c_{1} e^{r_{1}t} + c_{2} e^{r_{2}t}. \quad \text{nonumber}]
\]

The boundary conditions require that

\[
\begin{aligned}
[e_{1} &+ e^{r_{1}}t] + [c_{2} e^{r_{2}t}] \quad &\text{and} \quad [c_{1} e^{r_{1}}t] + [e_{2} e^{r_{2}t}] \quad &\text{and} \quad 0. \quad \text{end aligned}]
\end{aligned}
\]

Since the determinant of this system is \( (e^{r_{1}}t) - e^{r_{2}t}) \neq 0) \), the system has only the trivial solution. Therefore \( (\lambda) \) isn’t an eigenvalue of Equation \( \ref{eq:13.2.2} \).

If \( (\lambda = 1/4) \) then \( (r_{1} = r_{2} = -3/2) \), so the general solution of Equation \( \ref{eq:13.2.2} \) is

\[
[y = e^{-3x/2}(c_{1} + c_{2}x). \quad \text{nonumber}]
\]

The boundary condition \( (y(0) = 0) \) requires that \( (c_{1} = 0) \), so \( (y = c_{2}xe^{-3x/2}) \) and the boundary condition \( (y(0)) \) requires that \( (c_{2} = 0) \). Therefore \( (\lambda = 1/4) \) isn’t an eigenvalue of Equation \( \ref{eq:13.2.2} \).
If $\lambda > 1/4$, then
\[ r_1 = -\frac{3}{2} + i\omega \quad \text{and} \quad r_2 = -\frac{3}{2} - i\omega, \]
with
\[ \omega = \frac{\sqrt{4\lambda - 1}}{2} \quad \text{or equivalently} \quad \lambda = \frac{1 + 4\omega^2}{4}. \]

In this case the general solution of the differential equation in Equation \ref{eq:13.2.2} is
\[ y = e^{-3x/2}(c_1 \cos \omega x + c_2 \sin \omega x). \]

The boundary condition $y(0) = 0$ requires that $(c_1) = 0$, so $y = c_2 e^{-3x/2} \sin \omega x$, which holds with $(c_2) = 0$ if and only if $\omega = n\pi$, where $(n)$ is an integer. We may assume that $(n)$ is a positive integer. (Why?). From Equation \ref{eq:13.2.3}, the eigenvalues are $(\lambda_n) = (1 + 4n^2\pi^2)/4$, with associated eigenfunctions
\[ y_n = e^{-3x/2}\sin n\pi x, \quad n = 1, 2, 3, \ldots. \]

Example \(\PageIndex{2}\)

Solve the eigenvalue problem
\[ x^2y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0. \]

Solution

If $\lambda = 0$, the differential equation in Equation \ref{eq:13.2.4} reduces to $(x(xy')' = 0)$, so $(xy' = c_1)$,
\[ y = \frac{c_1}{x}, \quad \text{and} \quad y = c_1 \ln x + c_2. \]
The boundary condition $y(1) = 0$ requires that $(c_2) = 0$, so $(y = c_1 \ln x)$. The boundary condition $y(2) = 0$ requires that $(c_1 \ln 2) = 0$, so $(c_1) = 0$. Therefore zero isn’t an eigenvalue of Equation \ref{eq:13.2.4}.

If $\lambda < 0$, we write $\lambda = -k^2$ with $k > 0$, so Equation \ref{eq:13.2.4} becomes
\[ x^2y'' + xy' - k^2y = 0, \]
an Euler equation (Section 7.4) with indicial equation
\[ r^2 - k^2 = (r-k)(r+k) = 0. \]

Therefore
\[ y = c_1 x^k + c_2 x^{-k}. \]
The boundary conditions require that
\[
\begin{aligned}
\phantom{2^{k}}c_{1}+\phantom{2^{-k}}c_{2} &= 0 \\
2^{k}c_{1}+2^{-k}c_{2} &= 0.
\end{aligned}
\]
Since the determinant of this system is \(2^{-k}-2^{k}\ne0\), \((c_{1}=c_{2}=0)\). Therefore Equation \ref{eq:13.2.4} has no negative eigenvalues.

If \((\lambda>0)\) we write \((\lambda=k^{2}\{2\})\) with \((k>0)\). Then Equation \ref{eq:13.2.4} becomes
\[
[x^{2}\{2\}y''+xy' +k^{2}y=0, \nonumber \]

an Euler equation with indicial equation
\[
[r^{2}+k^{2}=(r-ik)(r+ik)=0, \nonumber \]

so
\[
[y=c_{\cdot1}\cos(k\ln x)+c_{\cdot2}\sin(k\ln x). \nonumber \]

The boundary condition \((y(1)=0)\) requires that \((c_{\cdot1}=0)\). Therefore \((y=c_{\cdot2}\sin(k\ln x))\). This holds with \((c_{\cdot2}\ne0)\)
if and only if \((k=n\pi/\ln 2)\), where \((n)\) is a positive integer. Hence, the eigenvalues of Equation \ref{eq:13.2.4} are
\[
(\lambda_{n}=n\pi/\ln2)^{2}, \text{ with associated eigenfunctions}
\]
\[
y_{n}=\sin\left(\frac{n\pi}{\ln2}\ln x\right), \quad n=1,2,3,\ldots. \nonumber \]

For theoretical purposes, it is useful to rewrite the differential equation in Equation \ref{eq:13.2.1} in a different form, provided by the next theorem.

Theorem \(\PageIndex{1}\)

If \((P_{0}, P_{1}, P_{2}, \text{ and } R)\) are continuous and \((P_{0})\) and \((R)\) are positive on a closed interval \([a,b]\), then the equation
\[
P_{0}(x)y''+P_{1}(x)y'+P_{2}(x)y+\lambda R(x)y=0
\]
can be rewritten as
\[
(p(x)y')'+q(x)y+\lambda r(x)y=0, \nonumber \]

where \((p), (p'), (q)\) and \((r)\) are continuous and \((p)\) and \((r)\) are positive on \([a,b]\).

Proof

We begin by rewriting Equation \ref{eq:13.2.5} as
\[
y''+u(x)y'+v(x)y+\lambda R_{1}(x)y=0, \nonumber \]

where \((u), (v)\) are continuous and \((v)\) and \((u)\) are positive on \([a,b]\).
with \(u=P_1/P_0\), \(v=P_2/P_0\), and \(R_1=R/P_0\). (Note that \(R_1\) is positive on \([a,b]\).) Now let \(p(x)=e^{U(x)}\), where \(U\) is any antiderivative of \(u\). Then \(p\) is positive on \([a,b]\) and, since \(U'=u\),

\[p'(x)=p(x)u(x),\]

is continuous on \([a,b]\). Multiplying Equation \ref{eq:13.2.7} by \(p(x)\) yields

\[p(x)y''+p(x)u(x)y'+p(x)v(x)y+\lambda p(x)R_1(x)y=0.\]

Since \(p\) is positive on \([a,b]\), this equation has the same solutions as Equation \ref{eq:13.2.5}. From Equation \ref{eq:13.2.8},

\[(p(x)y')'=p(x)y''+p(x)u(x)y',\nonumber\]

so Equation \ref{eq:13.2.9} can be rewritten as in Equation \ref{eq:13.2.6}, with \(q(x)=p(x)v(x)\) and \(r(x)=p(x)R_1(x)\). This completes the proof.

It is to be understood throughout the rest of this section that \(p\), \(q\), and \(r\) have the properties stated in Theorem \(\PageIndex{1}\). Moreover, whenever we write \(Ly\) in a general statement, we mean

\[Ly=(p(x)y')'+q(x)y.\nonumber\]

The differential equation Equation \ref{eq:13.2.6} is called a Sturm-Liouville equation, and the eigenvalue problem

\[\text{\ref{eq:13.2.10}}, \quad \text{\ref{eq:13.2.1}}, \quad B_{1}(y)=0, \quad B_{2}(y)=0,\]

which is equivalent to Equation \ref{eq:13.2.1}, is called a Sturm-Liouville problem.

Example \(\PageIndex{3}\)

Rewrite the eigenvalue problem

\[\text{\ref{eq:13.2.11}}, \quad \text{\ref{eq:13.2.1}}, \quad y''+3y'+(2+\lambda)y=0, \quad y(0)=0, \quad y(1)=0,\]

of Theorem \(\PageIndex{1}\) as a Sturm-Liouville problem.

Solution

Comparing Equation \ref{eq:13.2.11} to Equation \ref{eq:13.2.7} shows that \(u(x)=3\), so we take \(U(x)=3x\) and \(p(x)=e^{3x}\). Multiplying the differential equation in Equation \ref{eq:13.2.11} by \(e^{3x}\) yields

\[e^{3x}(y''+3y')+2e^{3x}y+\lambda e^{3x}y=0.\nonumber\]

Since
\(e^{3x}(y''+3y')=(e^{3x}y')', \text{ nonumber}\)

Equation \ref{eq:13.2.11} is equivalent to the Sturm–Liouville problem

\[\begin{align*}
\text{Example } \PageIndex{4} & \quad \text{Rewrite the eigenvalue problem} \n\end{align*}\]

\[\begin{align*}
\text{Example } \PageIndex{5} & \quad \text{Solve the Sturm–Liouville problem} \n\end{align*}\]


\[\text{\label{eq:13.2.15}} y''+\lambda y=0, \quad y(0)+y'(0)=0, \quad y(1)+3y'(1)=0.\]

**Solution**

If \(\lambda=0\), the differential equation in Equation \ref{eq:13.2.15} reduces to \(y''=0\), with general solution \(y=c_1+c_2x\). The boundary conditions require that

\[\begin{aligned}
    c_1+c_2 &= 0 \\
    c_1+4c_2 &= 0,
\end{aligned}\]

so \(c_1=c_2=0\). Therefore zero isn’t an eigenvalue of Equation \ref{eq:13.2.15}.

If \(\lambda<0\), we write \(\lambda=-k^2\) where \(k>0\), and the differential equation in Equation \ref{eq:13.2.15} becomes \(y''-k^2y=0\), with general solution

\[\text{\label{eq:13.2.16}} y=c_1\cosh kx+c_2\sinh kx,\]

so

\[y'=k(c_1\sinh kx+c_2\cosh kx). \nonumber\]

The boundary conditions require that

\[\text{\label{eq:13.2.17}} \begin{array}{c}
    c_1+kc_2=0 \\
    (\cosh k+3k\sinh k)c_1+(\sinh k+3k\cosh k)c_2=0
\end{array}\]

The determinant of this system is

\[\begin{aligned}
    D_N(k) &= \left|\begin{array}{cc}
        1 & k \\
        \cosh k+3k\sinh k & \sinh k+3k\cosh k
    \end{array}\right| \\
    &= (1-3k^2)\sinh k+2k\cosh k.
\end{aligned}\]

Therefore the system Equation \ref{eq:13.2.17} has a nontrivial solution if and only if \(D_N(k)=0\) or, equivalently,

\[\text{\label{eq:13.2.18}} \tanh k=-\frac{2k}{1-3k^2}.\]

The graph of the right side (Figure \ref{PageIndex{1}}) has a vertical asymptote at \(k=1/\sqrt{3}\). Since the two sides have different signs if \(k<1/\sqrt{3}\), this equation has no solution in \((0,1/\sqrt{3})\). Figure \ref{PageIndex{1}} shows the graphs of the two sides of Equation \ref{eq:13.2.18} on an interval to the right of the vertical asymptote, which is indicated by the dashed line. You can see that the two curves intersect near \(k=0.12\). Given this estimate, you can use Newton’s to compute \(k_0\) more accurately. We computed \((k_0)\approx 1.219395\). Therefore \((-k_0)^2\approx -1.2587483\) is an eigenvalue of Equation \ref{eq:13.2.18}. From Equation \ref{eq:13.2.16} and the first equation in Equation \ref{eq:13.2.17},

\[y_0=k_0\cosh k_0x-k_0\sinh k_0x. \nonumber\]
If \(\lambda > 0\) we write \(\lambda = k^2\) where \(k > 0\), and differential equation in Equation \ref{eq:13.2.15} becomes \((y'' + k^2 y = 0)\), with general solution

\[
y = \cos kx + c_2 \sin kx,
\]

so

\[
y' = k(-c_1 \sin kx + c_2 \cos kx).
\]

The boundary conditions require that

\[
\begin{array}{c}
c_1 + kc_2 = 0 \\
(c_1 \cos k - 3k \sin k)c_1 + (\sin k + 3k \cos k)c_2 = 0,
\end{array}
\]

The determinant of this system is

\[
D_P(k) = 
\begin{vmatrix}
1 & k \\
\cos k - 3k \sin k & \sin k + 3k \cos k
\end{vmatrix}
= (1 + 3k^2) \sin k + 2k \cos k.
\]

The system Equation \ref{eq:13.2.20} has a nontrivial solution if and only if \((D_P(k) = 0)\) or, equivalently,

\[
\tan k = \frac{2k}{1 + 3k^2}.
\]

Figure \(\PageIndex{2}\) shows the graphs of the two sides of this equation. You can see from the figure that the graphs intersect at infinitely many points \((k \approx n \pi)\) \((n = 1), (2), (3), \ldots\), where the error in this approximation approaches zero as \(n \to \infty\). Given this estimate, you can use Newton’s method to compute \((k_n)\) more accurately. We computed

\[
\begin{aligned}
k_1 &\approx 2.9256856, \\
k_2 &\approx 6.1765914, \\
k_3 &\approx 9.3538959, \\
k_4 &\approx 12.5132570.
\end{aligned}
\]

The estimates of the corresponding eigenvalues \((\lambda_n = k_n^2)\) are
From Equation \ref{eq:13.2.19} and the first equation in Equation \ref{eq:13.2.20},
\[y_{n}=k_{n}\cos k_{n}x-\sin k_{n}x\nonumber\]
is an eigenfunction associated with \(\lambda_{n}\).

Since the differential equations in Equation \ref{eq:13.2.12} and Equation \ref{eq:13.2.14} are more complicated than those in Equation \ref{eq:13.2.11} and Equation \ref{eq:13.2.13} respectively, what is the point of Theorem \ref{PageIndex1}? The point is this: to solve a specific problem, it may be better to deal with it directly, as we did in Examples \ref{PageIndex1} and \ref{PageIndex2}; however, we’ll see that transforming the general eigenvalue problem Equation \ref{eq:13.2.1} to the Sturm–Liouville problem Equation \ref{eq:13.2.10} leads to results applicable to all eigenvalue problems of the form Equation \ref{eq:13.2.1}.

Theorem \ref{PageIndex2}

If
\[Ly=(p(x)y')'+q(x)y\nonumber\]
and \(u\) and \(v\) are twice continuously functions on \([a,b]\) that satisfy the boundary conditions \((B_{1}(y)=0)\) and \((B_{2}(y)=0)\) then
\[\label{eq:13.2.21}\int_{a}^{b}[u(x)Lv(x)-v(x)Lu(x)]dx=0.\]

**Proof**

Integration by parts yields
\[\begin{aligned} \int_{a}^{b}[u(x)Lv(x)-v(x)Lu(x)]\,dx &= \int_{a}^{b}[u(x)(p(x)v'(x))'-v(x)(p(x)u'(x))']\,dx \\ &= p(x)[u(x)v'(x)-u'(x)v(x)]\bigg|_{a}^{b} &- \int_{a}^{b}p(x)[u'(x)v'(x)-u'(x)v'(x)]\,dx. \end{aligned}\]

Since the last integral equals zero,

\[\int_{a}^{b}[u(x)Lv(x)-v(x)Lu(x)]\,dx = p(x)[u(x)v'(x)-u'(x)v(x)]\bigg|_{a}^{b}.\]

By assumption, \(B_{1}(u)=B_{1}(v)=0\) and \(B_{2}(u)=B_{2}(v)=0\). Therefore

\[\begin{aligned} \alpha u(a)+\beta u'(a)&=0 \\ \alpha v(a)+\beta v'(a)&=0 \quad \text{and} \quad \begin{gathered} \rho u(b)+\delta u'(b)=0 \\ \rho v(b)+\delta v'(b)=0. \end{gathered} \]

Since \(\alpha^2+\beta^2>0\) and \(\rho^2+\delta^2>0\), the determinants of these two systems must both be zero; that is,

\[\begin{align*} 
[u(x)v'(x)-u'(x)v(x) &= u(b)v'(b)-u'(b)v(b)=0. \nonumber 
\end{align*}\]

This and Equation \ref{eq:13.2.22} imply Equation \ref{eq:13.2.21}, which completes the proof.

The next theorem shows that a Sturm–Liouville problem has no complex eigenvalues.

**Theorem \PageIndex{3}**

If \(\lambda=p+qi\) with \(q\neq0\) then the boundary value problem

\[Ly+\lambda r(x)y=0, \quad B_{1}(y)=0, \quad B_{2}(y)=0 \nonumber \]

has only the trivial solution.

**Proof**

For this theorem to make sense, we must consider complex-valued solutions of

\[\label{eq:13.2.23} Ly+(p+iq)r(x,y)y=0.\]

If \(y=u+iv\) where \(u\) and \(v\) are real-valued and twice differentiable, we define \(y'=u'+iv'\) and \(y''=u''+iv''\). We say that \(y\) is a solution of Equation \ref{eq:13.2.23} if the real and imaginary parts of the left side of Equation \ref{eq:13.2.23} are both zero. Since \((Ly=(p(x)'y')'+q(x)y')\) and \((p), \quad (q)\), and \((r)\) are real-valued,

\[\begin{aligned} Ly+\lambda r(x,y)&=L(u+iv)+(p+iq)r(x)(u+iv) \\ &=Lu+r(x)(pu-qv)+i[Lv+r(x)(pu+qv)], \end{aligned}\]

so \((Ly+\lambda r(x,y)y=0)\) if and only if

\[\begin{aligned} Ly+\lambda r(x)(pu-qv)&=0 \\ Ly+\lambda r(x)(qu+pv)&=0. \end{aligned}\]

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Multiplying the first equation by \((v)\) and the second by \((u)\) yields
\[
\begin{aligned}
vLu + r(x)(puv - qv^2) &= 0, \\
uLv + r(x)(qu^2 + puv) &= 0.
\end{aligned}
\]
Subtracting the first equation from the second yields
\[
[uLv - vLu + qr(x)(u^2 + v^2)] = 0, 
\]
so
\[
\int_{a}^{b}[u(x)Lv(x) - v(x)Lu(x)]\,dx + \int_{a}^{b}r(x)[u^2(x) + v^2(x)]\,dx = 0.
\]
Since
\[
[B_{1}(y) = B_{1}(u+iv) = B_{1}(u) + iB_{1}(v), 
\]
and
\[
[B_{2}(y) = B_{2}(u+iv) = B_{2}(u) + iB_{2}(v), 
\]
\((B_{1}(y) = 0)\) and \((B_{2}(y) = 0)\) implies that
\[
[B_{1}(u) = B_{2}(u) = B_{1}(v) = B_{2}(v) = 0, 
\]
Therefore Theorem \((\PageIndex{2})\) implies that first integral in Equation \ref{eq:13.2.24} equals zero, so Equation \ref{eq:13.2.24} reduces to
\[
[q\int_{a}^{b}r(x)[u^2(x) + v^2(x)]\,dx = 0. 
\]
Since \((r)\) is positive on \([a,b]\) and \((q\neq 0)\) by assumption, this implies that \((u\equiv 0)\) and \((v\equiv 0)\) on \([a,b]\).
Therefore \((y\equiv 0)\) on \([a,b]\), which completes the proof.

Theorem \((\PageIndex{4})\)

If \((\lambda_1)\) and \((\lambda_2)\) are distinct eigenvalues of the Sturm–Liouville problem
\[
[Ly + \lambda r(x)y = 0, \quad B_{1}(y) = 0, \quad B_{2}(y) = 0]
\]
with associated eigenfunctions \((u)\) and \((v)\) respectively then
\[
\int_{a}^{b}r(x)u(x)v(x)\,dx = 0. 
\]

**Proof**

Since \((u)\) and \((v)\) satisfy the boundary conditions in Equation \ref{eq:13.2.25}, Theorem \((\PageIndex{2})\) implies that
Since \( (\text{Lu}=-\lambda_1 ru) \) and \( (\text{Lv}=-\lambda_2 rv) \), this implies that
\[
\int_{a}^{b}[u(x)\text{Lv}(x)-v(x)\text{Lu}(x)]\,dx=0.
\]
Since \( (\lambda_1\ne\lambda_2) \), this implies Equation \ref{eq:13.2.26}, which completes the proof.

If \( (u) \) and \( (v) \) are any integrable functions on \([a,b]\) and
\[
\int_{a}^{b} r(x)u(x)v(x)\,dx=0,
\]
we say that \( (u) \) and \( (v) \) orthogonal on \([a,b]\) with respect to \( (r=r(x)) \).

Theorem 13.1.1 implies the next theorem.

Theorem \( \PageIndex{5} \)

If \( (u)\not\equiv0 \) and \( (v) \) both satisfy
\[
[Ly+\lambda r(x)y=0, \quad B_1(y)=0, \quad B_2(y)=0, \quad ]
\]
then \( (v=cu) \) for some constant \( (c) \).

We’ve now proved parts of the next theorem. A complete proof is beyond the scope of this book.

Theorem \( \PageIndex{6} \)

The set of all eigenvalues of the Sturm–Liouville problem
\[
[Ly+\lambda r(x)y=0, \quad B_1(y)=0, \quad B_2(y)=0 \quad ]
\]
can be ordered as
\[
[\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots, \quad ]
\]
and
\[
[\lim_{n\to\text{infty}} \lambda_n=\text{infty} \quad ]
\]
For each \( (n) \) if \( (y_{\{n\}}) \) is an arbitrary \( (\lambda_{\{n\}}) \)-eigenfunction, then every \( (\lambda_{\{n\}}) \)-eigenfunction is a constant multiple of \( (y_{\{n\}}) \). If \( (m\ne n\) \) \( (y_{\{m\}}) \) and \( (y_{\{n\}}) \) are orthogonal \((a,b]\) with respect to \( (r=r(x)) \) that is,
\[
\int_{a}^{b} r(x)y_{\{m\}}(x)y_{\{n\}}(x)\,dx=0.
\]
You may want to verify Equation \ref{eq:13.2.27} for the eigenfunctions obtained in Examples \ref{PageIndex{1}} and \ref{PageIndex{2}}.

In conclusion, we mention the next theorem. The proof is beyond the scope of this book.

Theorem \ref{PageIndex{7}}

Let \(\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots\) be the eigenvalues of the Sturm–Liouville problem

\[
Ly + \lambda r(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0.
\]

with associated eigenvectors \(y_1, y_2, \ldots, y_n, \ldots\). Suppose \(f\) is piecewise smooth Theorem 11.2.3 on \([a,b]\). For each \(n\), let

\[
c_n = \frac{\int_{a}^{b} r(x)f(x)y_n(x) \, dx}{\int_{a}^{b} r(x)y_n^2(x) \, dx}.
\]

Then

\[
\frac{f(x-)+f(x+)}{2} = \sum_{n=1}^{\infty} c_n y_n(x)
\]

for all \(x\) in the open interval \((a,b)\).