1.10: Logical Implication

At first glance it seems that a large portion of mathematics can be broken down into answering questions of the form: If I know this statement is true, is it necessarily the case that this other statement is true? In this section we will formalize that question.

**Definition 1.9.1.** Suppose that \( \Delta \) and \( \Gamma \) are sets of \( \mathcal{L} \)-formulas. We will say that \( \Delta \) **logically implies** \( \Gamma \) and write \( \Delta \models \Gamma \) if for every \( \mathcal{L} \)-structure \( \mathfrak{A} \), if \( \mathfrak{A} \models \Delta \), then \( \mathfrak{A} \models \Gamma \).

This definition is a little bit tricky. It says that if \( \Delta \) is true in \( \mathfrak{A} \), then \( \Gamma \) is true in \( \mathfrak{A} \). Remember, for \( \Delta \) to be true in \( \mathfrak{A} \), it must be the case that \( \mathfrak{A} \models \Delta \left[ s \right] \) for every assignment function \( s \). See Exercise 4.

If \( \Gamma = \{ \gamma \} \) is a set consisting of a single formula, we will write \( \Delta \models \gamma \) rather than the official \( \Delta \models \{ \gamma \} \).

**Definition 1.9.2.** An \( \mathcal{L} \)-formula \( \phi \) is said to be **valid** if \( \emptyset \models \phi \), in other words, if \( \phi \) is true in every \( \mathcal{L} \)-structure with every assignment function \( s \). In this case, we will write \( \models \phi \).

*Chaff:* It doesn't seem like it would be easy to check whether \( \Delta \models \Gamma \). To do so directly would mean that we would have to examine every possible \( \mathcal{L} \)-structure and every possible assignment function \( s \), of which there will be many.

I'm also sure that you've noticed that this double turnstyle symbol, \( \models \), is getting a lot of use. Just remember that if there is a structure on the left, \( \mathfrak{A} \models \gamma \), we are discussing truth in a single structure. If
there is a set of sentences on the left, $\Gamma \models \sigma$, then we are discussing logical implication.

**Example 1.9.3.** Let $\mathcal{L}$ be the language consisting of a single binary relation symbol, $P$, and let $\sigma$ be the sentence $\left( \exists y \forall x \ P( x, y \text{ right}) \right) \rightarrow \left( \forall x \exists y \ P( x, y \text{ right}) \right)$. We show that $\Gamma \models \sigma$ is valid.

So let $\mathfrak{A}$ be any $\mathcal{L}$-structure and let $s : Vars \rightarrow A$ be any assignment function. We must show that

$$\mathfrak{A} \models \left( \exists y \forall x \ P( x, y \text{ right}) \right) \rightarrow \left( \forall x \exists y \ P( x, y \text{ right}) \right).$$

Assume that $\mathfrak{A} \models \left( \exists y \forall x \ P( x, y \text{ right}) \right)$, we know that there is an element of the universe, $\mathcal{A}$, such that $\mathfrak{A} \models \forall x \ P( x, y \text{ right})$. And so, again by the definition of satisfaction, we know that if $\mathfrak{A} \models \left( b \right)$ is any element of $\mathcal{A}$, $\mathfrak{A} \models \forall x \ P( x, y \text{ right})$. If we chase through the definition of satisfaction (Definition 1.7.4) and of the various assignment functions, this means that for our one fixed $\left( a \right)$, the ordered pair $\left( b, a \right) \in P^A$ for any choice of $\left( b \in A \right)$.

We have to prove that $\mathfrak{A} \models \left( \forall x \exists y \ P( x, y \text{ right}) \right)$. As the statement of interest is universal, we must show that, if $\mathfrak{A} \models \left( \forall c \ P( x, y \text{ right}) \right)$, which means that we must produce an element of the universe, $\mathcal{A}$, such that $\mathfrak{A} \models \forall x \ P( x, y \text{ right})$. Again, from the definition of satisfaction this means that we must find a $\left( d \in A \right)$ such that $\mathfrak{A} \models \left( \left( c, d \right) \in P^\mathfrak{A} \right)$. Fortunately, we have such a $\left( d \right)$ in hand, namely $\left( a \right)$. As we know, $\mathfrak{A} \models \left( a, d \right) \in P^\mathfrak{A}$, we have shown $\mathfrak{A} \models \left( \forall x \exists y \ P( x, y \text{ right}) \right)$, and we are finished.

**Exercises**

1. Show that $\models \left( \alpha \rightarrow \beta \right)$ for any formulas $\alpha$ and $\beta$. Translate this result into everyday English. Or Norwegian, if you prefer.

2. Show that the formula $\left( \forall x \ P( x \text{ right}) \right)$ is valid. Show that the formula $\left( \forall y \ P( x \text{ right}) \right)$ is not valid. What can you prove about the formula $\left( \neg x = y \right)$ in terms of validity?

3. Suppose that $\left( \phi \right)$ is an $\mathcal{L}$-formula and $\left( x \right)$ is a variable. Prove that $\left( \phi \right)$ is valid if and only if $\left( \forall x \phi \right)$ is valid. Thus, if $\left( \phi \right)$ has free variables $\left( x \right)$, $\left( y \right)$, and $\left( z \right)$, $\left( \phi \right)$ will be valid if and only if $\left( \forall x \forall y \forall z \phi \right)$ is valid. The sentence $\left( \forall x \forall y \forall z \phi \right)$ is called the universal closure of $\left( \phi \right)$.

4. (a) Assume that $\left( \models \phi \right)$, show that $\left( \models \psi \right)$. (b) Suppose that $\left( \phi \right)$ is $\left( x < y \right)$ and $\left( \psi \right)$ is $\left( x < y \right)$. Show that $\left( \phi \models \psi \right)$ but $\left( \models \phi \right)$ does not imply $\left( \models \psi \right)$.

[This exercise shows that the two possible ways to define logical equivalence are not equivalent. The strong form of the definition says that $\left( \phi \right)$ and $\left( \psi \right)$ are logically equivalent if $\left( \models \phi \right)$ and $\left( \models \psi \right)$ are the strong form of the definition states that $\left( \phi \right)$ and $\left( \psi \right)$ are logically equivalent if $\left( \models \phi \right)$ and $\left( \models \psi \right)$ are logically equivalent.]
logically equivalent if \( \phi \models \psi \) and \( \psi \models \phi \).]